CHAPTER 23

The Ultrastable System

23/1. The definition and description already given in S. 8/6 and 7 have established the elementary properties of the ultrastable system. A restatement in mathematical form, however, has the advantage of rendering a misunderstanding less likely, and of providing a base for quantitative studies.

If a system is ultrastable, it is composed of main variables x_i and of step-functions a_i, so that the whole is absolute:

$$\frac{dx_i}{dt} = f_i(x\,;\ a) \qquad (i = 1,\ \ldots,\ n)$$

$$\frac{da_i}{dt} = g_i(x\,;\ a) \qquad (i = 1,\ 2,\ \ldots)$$

The functions g_i must be given some form like that of S. 22/2. The system is started with the representative point within the critical surface $\phi(x) = 0$, contact with which makes the step-functions change value. When they change, the new values are to be random samples from some distribution, assumed given.

Thus in the homeostat, the equations of the main variables are (S. 19/11):

$$\frac{dx_i}{dt} = a_{i_1}x_1 + a_{i_2}x_2 + a_{i_3}x_3 + a_{i_4}x_4 \qquad (i = 1,\ 2,\ 3,\ 4)$$

The a's are step-functions, coming from a distribution of 'rectangular' form, lying evenly between -1 and $+1$. The critical surfaces of the a's are specified approximately by $|x| \pm \frac{\pi}{4} = 0$. Each individual step-function a_{jk} depends only on whether x_j crosses the critical surface.

As the a's change discontinuously, an analytic integration of the differential equations is not, so far as I am aware, possible. But the equations, the description, and the schedule of the uniselector-wirings (the random samples) define uniquely the behaviour of the x's and the a's. So the behaviour could be

computed to any degree of accuracy by a numerical method. The proof given in Chapter 8, though verbal, is adequate to establish the elementary properties of the system. A rigorous statement and proof would add little of real value.

23/2. How many trials will be necessary, on the average, for a terminal field to be found? If an ultrastable system has a probability p that a new field of the main variables will be stable, and if the fields' probabilities are independent, then the number of fields occurring (including the terminal) will be, on the average, $1/p$.

For at the first field, a proportion p will be terminal, and $q\ (= 1 - p)$ will not. Of the latter, at the second field, the proportion p will be terminal and q not; so the total proportion stable at the second field will be pq, and the number still unstable q^2. Similarly the proportion becoming terminal at the u-th field will be pq^{u-1}. So the average number of trials made will be

$$\frac{p + 2pq + 3pq^2 + \ldots + upq^{u-1} + \ldots}{p + pq + pq^2 + \ldots + pq^{u-1} + \ldots} = \frac{1}{p}.$$

23/3. In an ultrastable system, a field may be terminal and yet show little resemblance to the 'normal' equilibrium which is necessary if the system is to show, after each of a variety of displacements, a return to the resting state. A field, for instance, might have a resting state at which only a single line of behaviour terminated: if the representative point were on that line the field would be terminal; but hardly any displacement would be followed by a return to the resting state.

It can, however, be shown that if a proportion of the fields evoked by the step-function changes are of this or similar type, then the terminal fields will contain them in smaller proportion. For, given a field and a closed critical surface, let k_1 be the proportion of lines of behaviour crossing the boundary which are stable. Thus in Figure 8/7/1, in I $k_1 = 0$, in II $k_1 = 0$, in III $k_1 = \frac{1}{2}$ approximately, and in IV $k_1 = 1$. To count the lines, the boundary surface could be divided into portions of equal area, small enough so that stable and unstable lines do not pass through the same area. Then if we assume that in any field the representative point is equally likely to start at any of the small areas, a field's chance of being terminal is proportional to k_1.

It follows that if the changes of step-functions evoke fields whose values of k_1 are distributed so that the probability of a field having a k_1-value between k_1 and $k_1 + dk_1$ is $\psi(k_1)dk_1$, then in the terminal fields the probablity is

$$\frac{k_1\psi(k_1)dk_1}{\int_0^1 k_1\psi(k_1)dk_1} \quad . \quad . \quad . \quad . \quad (1)$$

Figure 23/3/1 shows a possible distribution of values of k_1 in

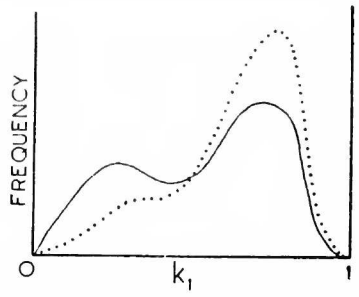

FIGURE 23/3/1 : Solid line : a distribution $\psi(k_1)$; broken line : the corresponding distribution $k_1\psi(k_1)$.

the original fields (solid line), and how k_1 would then be distributed in the terminal fields (broken line). The shift towards the higher values of k_1 is clear.

Fields with a low value of k_1, unsatisfactory for adaptation, tend therefore not to be terminal.

23/4. It was noticed in S. 13/4 that fields like A and B of Figure 13/4/1, though terminal, are defective in their persistence after small random disturbances. This idea may be given more precision.

Assume that the small random disturbances cause displacements which have some definite probability distribution, Gaussian say, so that if applied to the representative point when it is at some definite position in the field, there is a definite probability k_2 that a random displacement will not carry the point beyond the critical surface. Assume the representative point is always at the resting state or resting cycle. Then any terminal field has a unique value for k_2. If the field contains a single resting state, k_2 for that field is the probability, when the representative point

is at the resting state, that the application of a single random disturbance will not take the representative point beyond the critical surface. If the field has a resting cycle, k_2 is the average of the values when the representative point is on the many portions of the cycle, the value for each portion being weighted according to the time spent by the representative point in that portion. For more complex fields, k_2 could be defined, but a more detailed study is not necessary here.

Suppose that the ultrastable system, when the step-functions undergo random changes, yields terminal fields whose values of k_2 are distributed so that the proportion falling between k_2 and $k_2 + dk_2$ is $\phi(k_2)dk_2$. If to such fields, with k_2 lying between such limits, we apply one random disturbance, a proportion k_2 will not be changed; but the proportion $1 - k_2$ will be changed, and will be replaced by new terminal fields; their values of k_2 will be distributed again as $\phi(k_2)dk_2$, and this distribution will be added to that of the unchanged fields. In this way it is easy to show that the final distribution $\lambda(k_2)$ equals

$$A \frac{\phi(k_2)}{1 - k_2}$$

where A is a constant.

Examination of the form of the distribution $\lambda(k_2)$ shows that it is $\phi(k_2)$ heavily weighted in favour of the values of k_2 near 1. Such fields can only be those with the resting state or cycle near the centre of the region. So the result confirms the common-sense argument of S. 13/4. It will be noticed that the deduction is independent of the particular form of the distribution of disturbances.

References

Ashby, W. Ross. The physical origin of adaptation by trial and error. *Journal of general Psychology*, **32**, 13; 1945.

Idem. The nervous system as physical machine: with special reference to the origin of adaptive behaviour. *Mind*, **56**, 1; 1947.