CHAPTER 7

Step-Functions

7/1. SOMETIMES the behaviour of a variable (or parameter) can be described without reference to the cause of the behaviour: if we say a variable or system is a 'simple harmonic oscillator' the meaning of the phrase is well understood. Here we shall be more interested in the extent to which a variable displays constancy. Four types may be distinguished, and are illustrated in

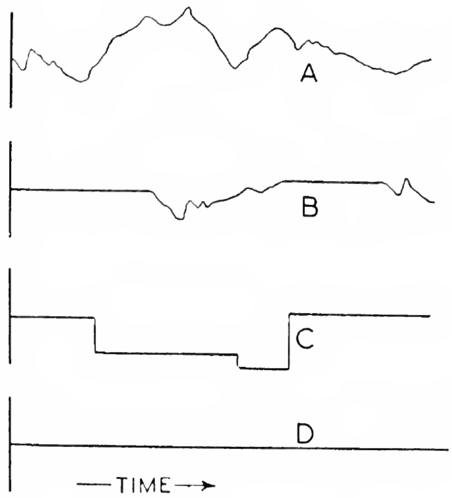

FIGURE 7/1/1: Types of behaviour of a variable: *A*, the full-function; *B*, the part-function; *C*, the step-function; *D*, the null-function.

Fig. 7/1/1. (*A*) The **full-function** has no finite interval of constancy; many common physical variables are of this type: the height of the barometer, for instance. (*B*) The **part-function** has finite intervals of change and finite intervals of constancy; it will be considered more fully in S. 14/12. (*C*) The **step-function** has finite intervals of constancy separated by instantaneous jumps.

And, to complete the set, we need (*D*) the **null-function**, which shows no change over the whole period of observation. The four types obviously include all the possibilities, except for mixed forms. The variables of Fig. 2/10/1 will be found to be part-, full-, step-, and null-, functions respectively.

In all cases the type-property is assumed to hold only over the period of observation: what might happen at other times is irrelevant.

Sometimes physical entities cannot readily be allotted their type. Thus, a steady musical note may be considered either as unvarying in intensity, and therefore a null-function, or as represented by particles of air which move continuously, and therefore a full-function. In all such cases the confusion is at once removed if one ceases to think of the real physical object with its manifold properties, and selects that variable in which one happens to be interested.

7/2. Step-functions occur abundantly in nature, though the very simplicity of their properties tends to keep them inconspicuous. 'Things in motion sooner catch the eye than what not stirs'. The following examples approximate to the step-function, and show its ubiquity:

(1) The electric switch has an *electrical resistance* which remains constant except when it changes by a sudden jump.

(2) The *electrical resistance* of a fuse similarly stays at a low value for a time and then suddenly changes to a very high value.

(3) The *viscosity* of water, measured as the temperature passes 0° C., changes similarly.

(4) If a piece of rubber is stretched, the pull it exerts is approximately proportional to its length. The *constant of proportionality* has a definite constant value unless the elastic is stretched so far that it breaks. When this happens the constant of proportionality suddenly becomes zero, i.e. it changes as a step-function.

(5) If a trajectory is drawn through the air, a few feet above the ground and parallel to it, the *resistance* it encounters as it meets various objects varies in step-function form.

(6) A stone, falling through the air into a pond and to the bottom, would meet *resistances* varying similarly.
(7) The *temperature* of a match when it is struck changes in step-function form.
(8) If strong acid is added in a steady stream to an unbuffered alkaline solution, the *p*H changes in approximately step-function form.
(9) If alcohol is added slowly with mixing to an aqueous solution of protein, the *amount of protein* precipitated changes in approximately step-function form.
(10) As the *p*H is changed, the *amount of adsorbed substance* often changes in approximately step-function form.
(11) By quantum principles, many *atomic* and *molecular variables* change in step-function form.
(12) The *blood flow* through the *ductus arteriosus*, when observed over an interval including the animal's birth, changes in step-function form.
(13) The *sex-hormone content* of the blood changes in step-function form as an animal passes puberty.
(14) Any variable which acts only in ' all or none ' degree shows this form of behaviour if each degree is sustained over a finite interval.

7/3. Few variables other than the atomic can change instantaneously; a more minute examination shows that the change is really continuous: the fusing of an electric wire, the closing of a switch, and the snapping of a piece of elastic. But if the event occurs in a system whose changes are appreciable only over some longer time, it may be treated without serious error as if it occurred instantaneously. Thus, if $x = \tanh t$, it will give a graph like A in Figure 7/3/1 if viewed over the interval from $t = -2$ to $t = +2$. But if viewed over the interval from $t = -40$ to $t = +40$, it would give a graph like B, and would approximate to the step-function form.

In any experiment, some ' order ' of the time-scale is always assumed, for the investigation never records both the very quick and the very slow. Thus to study a bee's honey-gathering flights, the observer records its movements. But he ignores the movement caused by each stroke of the wing: such movements are ignored as being too rapid. Equally, over an hour's experiment he ignores the fact that the bee at the end of the hour is a little older than it was at the beginning: this change is ignored as being too slow.

Such changes are eliminated by being treated as if they had their limiting values. If a single rapid change occurs, it is

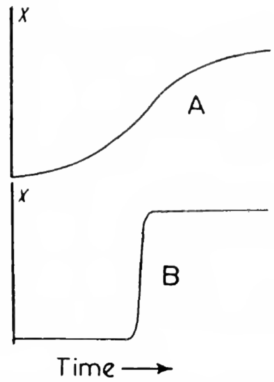

FIGURE 7/3/1: The same change viewed: (A) over one interval of time, (B) over an interval twenty times as long.

treated as instantaneous. If a rapid oscillation occurs, the variable is given its average value. If the change is very slow, the variable is assumed to be constant. In this way the concept of 'step-function' may legitimately be applied to real changes which are known to be not quite of this form.

7/4. Behaviour of step-function form is likely to be seen whenever we observe a 'machine' whose component parts are fast-acting. Thus, if we casually alter the settings of an unknown electronic machine we are not unlikely to observe, from time to time, sudden changes of step-function form, the suddenness being due to the speed with which the machine changes.

A reason can be given most simply by reference to Figure 4/3/1. Suppose that the curvature of the surface is controlled by a parameter which makes A rise and B fall. If the ball is resting at A, the parameter's first change will make no difference to the ball's lateral position, for it will continue to rest at A (though with lessened reaction if displaced.). As the parameter is changed further, the ball will continue to remain at A until A and B are level. Still the ball will make no movement. But if the parameter goes on changing and A rises above B, and if gravitation is intense and the ball fast-moving, then the ball will suddenly move to B. And here it will remain, however high A becomes and however low B. So, if the parameter changes steadily, the lateral position of the ball will tend to step-function form, approximating more closely as the passage of the ball for a given degree of slope becomes swifter.

The possibility need not be examined further, for no exact deductions will be drawn from it. The section is intended only to show that step-functions occur not uncommonly when the system under observation contains fast-acting components. The subject will be referred to again in S. 10/5.

Critical states

7/5. In any absolute system, the behaviour of a variable at any instant depends on the values which the variable and the others have at that instant (S. 2/15). If one of the variables behaves as a step-function the rule still applies : whether the variable remains constant or undergoes a change is determined both by the value of the variable and by the values of the other variables. So, given an absolute system with a step-function at a particular value, all the states with the step-function at that value can be divided into two classes : those whose occurrence does and those whose occurrence does not lead to a change in the step-function's value. The former are its **critical states** : should one of them occur, the step-function will change value. The critical state of an electric fuse is the number of amperes which will cause it to blow. The critical state of the ' constant of proportionality ' of an elastic strand is the length at which it breaks.

An example from physiology is provided by the urinary bladder when it has developed an automatic intermittently-emptying action after spinal section. The bladder fills steadily with urine, while at first the spinal centres for micturition remain inactive. When the volume of urine exceeds a certain value the centres become active and urine is passed. When the volume falls below a certain value, the centre becomes inactive and the bladder refills. A graph of the two variables would resemble Figure 7/5/1. The two-variable system is absolute, for it has the field of Figure 7/5/2. The variable y is approximately a step-function. When it is at 0, its critical state is $x = X_2$, $y = 0$, for the occurrence of this state

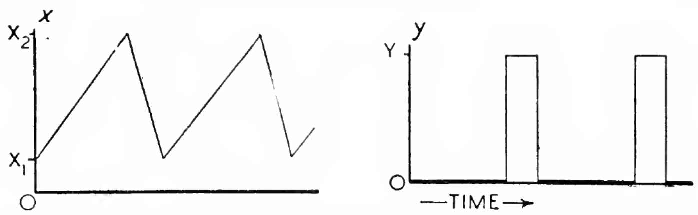

FIGURE 7/5/1 : Diagram of the changes in x, volume of urine in the bladder, and y, activity in the centre for micturition, when automatic action has been established after spinal section.

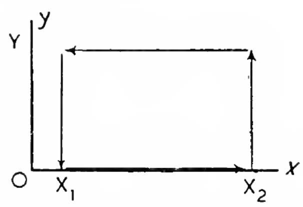

FIGURE 7/5/2 : Field of the changes shown in Figure 7/5/1.

determines a jump from 0 to Y. When it is at Y, its critical state is $x = X_1$, $y = Y$, for the occurrence of this state determines a jump from Y to 0.

7/6. A common, though despised, property of every machine is that it may ' break '. This event is in no sense unnatural, since it must follow the basic laws of physics and chemistry and is therefore predictable from its immediately preceding state. In general, when a machine ' breaks ' the representative point has met some critical state, and the corresponding step-function has changed value.

As is well known, almost any machine or physical system will break if its variables are driven far enough away from their usual values. Thus, machines with moving parts, if driven ever faster, will break mechanically ; electrical apparatus, if subjected to ever higher voltages or currents, will break in insulation ; machines made too hot will melt—if made too cold they may encounter other sudden changes, such as the condensation which stops a steam-engine from working below 100° C. ; in chemical dynamics, increasing concentrations may meet saturation, or may cause precipitation of proteins.

Although there is no rigorous law, there is nevertheless a widespread tendency for systems to show changes of step-function form if their variables are driven far from some usual value. Later (S. 10/2) it will be suggested that the nervous system is not exceptional in this respect.

Systems containing full- and null-functions

7/7. We shall now consider the properties shown by absolute systems that contain step-functions. But the discussion will be clearer and simpler if we first examine some simpler systems.

Suppose we have an absolute system composed wholly of full-functions and we ignore one of the variables. Every experimenter knows only too well what happens : the behaviour of the system becomes unpredictable. Every experimenter has spent time trying to make unpredictable experiments predictable ; he does it by identifying the unknown variable. The unknown variable may be scientifically trivial, like a loose screw, or important, like a co-enzyme in a metabolic system ; but in either case, he cannot establish a definite form of behaviour until he has identified and either controlled or observed the unknown variable. To ignore a *full*-function in an absolute system is to render the remainder non-absolute, so that no characteristic form of behaviour can be established.

On the other hand, an absolute system which includes null-functions may have the null-functions removed from it, or other null-functions added to it, and the new system will still be absolute. (The alteration is done, of course, not by interfering physically with the ' machine ', but by changing the list of variables.) Thus, if the two-variable system of the pendulum (S. 6/3) is absolute, and if the length of the pendulum stays constant once it is adjusted, then the system composed of the three variables :

(1) length of pendulum
(2) angular deviation
(3) angular velocity)

is also absolute. A formal proof is given in S. 21/4, but it follows readily from the definitions. (The reader should first verify that every null-function is itself an absolute system.) Conversely, if three variables A, B, N, are found to form an absolute system, and N is a null-function, then the system composed of A and B is absolute.

Unlike the full-function, then, the null-function may be omitted from a system, for its omission leaves the remainder still producing predictable behaviour.

Systems containing step-functions

7/8. Suppose that we have a system with three variables, A, B, S; that it has been tested and found absolute; that A and B are full-functions; and that S is a step-function. (Variables A and B, as in S. 21/3, will be referred to as **main** variables.) The phase-space of this system will resemble that of Figure 7/8/1 (a possible field has been sketched in). The phase-space no longer fills all three dimensions, but as S can take only discrete values, here assumed for simplicity to be a pair, the phase-space is restricted to two planes normal to S, each plane corresponding to a particular value of S. A and B being full-functions, the representative point will move on curves in each plane, describing a line of behaviour such as that drawn more heavily in the Figure. When

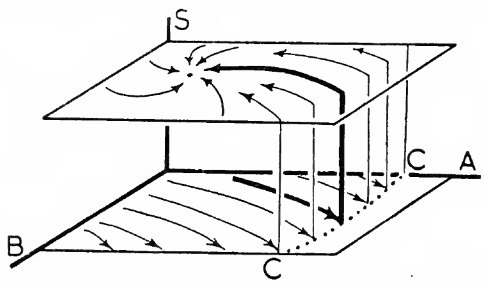

FIGURE 7/8/1: Field of an absolute system of three variables, of which S is a step-function. The states from C to C are the critical states of the step-function.

the line of behaviour meets the row of critical states at C—C, S jumps to its other value, and the representative point continues along the heavily marked line in the upper plane. In such a field the movement of the representative point is everywhere state-determined, for the number of lines from any point never exceeds one.

If, still dealing with the same real 'machine', we ignore S, and repeatedly form the field of the system composed of A and B, S being free to take sometimes one value and sometimes the other, we shall find that we get sometimes a field like I in Figure 7/8/2, and sometimes a field like II, the one or the other appearing according to the value that S happens to have at the time.

The behaviour of the system AB, in its apparent possession of two fields, should be compared with that of the system described in S. 6/3, where the use of two parameter-values also caused the appearance of two fields. But in the earlier case the change of the field was caused by the arbitrary action of the experimenter, who forced the parameter to change value, while in this case the change of the field of AB is caused by the inner mechanisms of the 'machine' itself.

The property may now be stated in general terms. Suppose, in an absolute system, that some of the variables are step-functions, and that these are ignored while the remainder (the main variables) are observed on many occasions by having their field constructed.

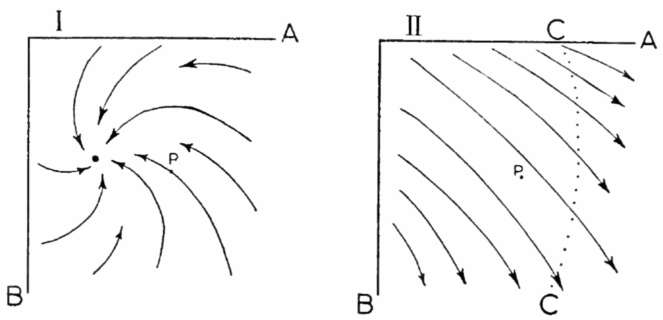

FIGURE 7/8/2 : The two fields of the system composed of A and B. P is in the same position in each field.

Then so long as no step-function changes value during the construction, the main variables will be found to form an absolute system, and to have a definite field. But on different occasions different fields may be found. *The number of different fields shown by the main variables is equal to the number of combinations of values provided by the step-functions.*

7/9. These considerations throw light on an old problem in the theory of mechanisms.

Can a 'machine' be at once determinate and capable of spontaneous change? The question would be contradictory if posed by one person, but it exists in fact because, when talking of living organisms, one school maintains that they are strictly determinate while another school maintains that they are capable of spontaneous change. Can the schools be reconciled?

The presence of step-functions in an absolute system enables both schools to be right, provided that those who maintain the determination are speaking of the system which comprises *all* the variables, while those who maintain the possibility of spontaneous change are speaking of the main variables only. For the whole system, which includes the step-functions, is absolute, has one field only, and is completely state-determined (like Figure 7/8/1). But the system of main variables may show as many different forms of behaviour (like Figure 7/8/2, I and II) as the step-functions possess combinations of values. And if the step-functions are not accessible to observation, the change of the main variables from one form of behaviour to another will seem to be spontaneous, for no change or state in the main variables can be assigned as its cause.

The argument may seem plausible, but it is stronger than that. It may be proved (S. 22/5) that if a ' machine ', known to be completely isolated and therefore absolute, produces several characteristic forms of behaviour, i.e. possesses several fields, then there *must* be, interacting with the observed variables and included within the ' machine ', some step-functions.