CHAPTER 4

Stability

4/1. THE words ' stability ', ' steady state ', and ' equilibrium ' are used by a variety of authors with a variety of meanings, though there is always the same underlying theme. As we shall be much concerned with stability and its properties, an exact definition must be provided.

The subject may be opened by a presentation of the three standard elementary examples. A cube resting with one face on a horizontal surface typifies ' stable ' equilibrium ; a sphere resting on a horizontal surface typifies ' neutral ' equilibrium ; and a cone balanced on its point typifies ' unstable ' equilibrium. With neutral and unstable equilibria we shall have little concern, but the concept of ' stable equilibrium ' will be used repeatedly.

These three dynamic systems are restricted in their behaviour by the fact that each system contains a fixed quantity of energy, so that any subsequent movement must conform to this invariance. We, however, shall be considering systems which are abundantly supplied with free energy so that no such limitation is imposed. Here are two examples.

The first is the Watt's governor. A steam-engine rotates a pair of weights which, as they are rotated faster, separate more widely by centrifugal action ; their separation controls mechanically the position of the throttle ; and the position of the throttle controls the flow of steam to the engine. The connections are arranged so that an increase in the speed of the engine causes a decrease in the flow of steam. The result is that if any transient disturbance slows or accelerates the engine, the governor brings the speed back to the usual value. By this return the system demonstrates its stability.

The second example is the thermostat, of which many types exist. All, however, work on the same principle : a chilling of the bath causes a change which in its turn causes the heating to become more intense or more effective ; and vice versa. The

result is that if any transient disturbance cools or overheats the bath, the thermostat brings the temperature back to the usual value. By this return the system demonstrates its stability.

4/2. An important feature of stability is that it does not refer to a material body or ' machine ' but only to some aspect of it. This statement may be proved most simply by an example showing that a single material body can be in two different equilibrial states at the same time. Consider a square card balanced exactly on one edge : to displacements at right angles to this edge the card is unstable ; to displacements exactly parallel to this edge it is, theoretically at least, stable.

The example supports the thesis that we do not, in general, study physical bodies but only entities carefully abstracted from them. The concept of stability must therefore be defined in terms of the basic primary operations (S. 2/3).

4/3. Consider next a corrugated surface, laid horizontally, with a ball rolling from a ridge down towards a trough. A photograph taken in the middle of its roll would look like Figure 4/3/1. We might think of the ball as being unstable because it has rolled away from the ridge, until we realise that we can also think of it as stable because it is rolli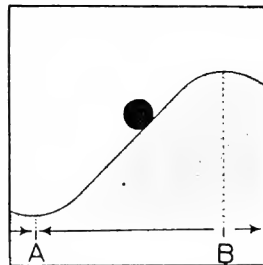ng towards the trough. The duality shows we are approaching the concept in the wrong way. The situation can be made clearer if we remove the ball and consider only the surface. The top of the ridge, as it would affect the roll of a ball, is now recognised as a position of unstable equilibrium, and the bottom of the trough as a position of stability. We now see that, if friction is sufficiently marked for us to be able to neglect momentum, the system composed of the single variable ' distance of the ball laterally ' is absolute and has a definite, permanent field, which is sketched in the Figure.

FIGURE 4/3/1.

From *B* the lines of behaviour diverge, but to *A* they converge. We conclude tentatively that the concept of ' stability ' belongs not to a material body but to a field. It is shown by a field if the lines of behaviour converge. (An exact definition is given in S. 4/8.)

4/4. This preliminary remark begins to justify the emphasis placed on absoluteness. Since stability is a feature of a field, and since only regular systems have unchanging fields (S. 19/16) it follows that to discuss stability in a system we must suppose that the system is regular : we cannot test the stability of a thermostat if some arbitrary interference continually upsets it.

But regularity in the system is not sufficient. If a field had lines criss-crossing like those of Figure 2/15/2 we could not make any simple statement about them. Only when the lines have a smooth flow like those of Figures 4/5/1, 4/5/2 or 4/10/1 can a simple statement be made about them. And this property implies (S. 19/12) that the system must be absolute.

4/5. To illustrate that the concept of stability belongs to a field, let us examine the fields of the previous examples.

The cube resting on one face yields an absolute system which has two variables :

(x) the angle which the face makes with the horizontal, and

(y) the rate at which this angle changes.

(This system allows for the momentum of the cube.) If the cube does not bounce when the face meets the table, the field is similar
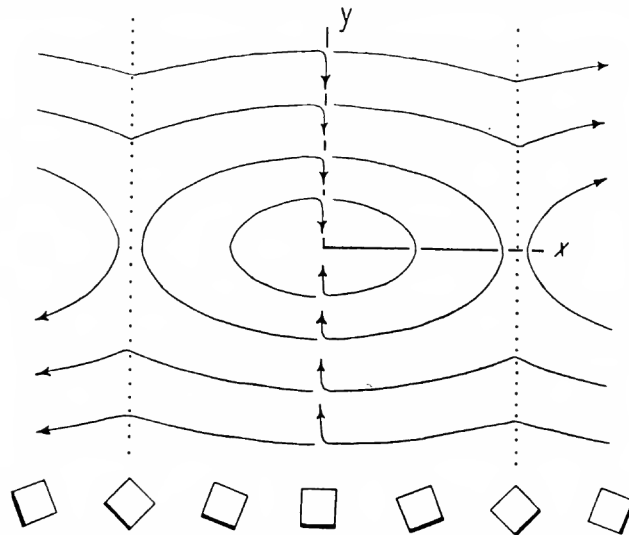

FIGURE 4/5/1 : Field of the two-variable system described in the text. Below is shown the cube as it would appear in elevation when its main face, shown by a heavier line, is tilted through the angle x.

to that sketched in Figure 4/5/1. The stability of the cube when resting on a face corresponds in the field to the convergence of the lines of behaviour to the centre.

The square card balanced on its edge can be represented approximately by two variables which measure displacements at right

angles (x) and parallel (y) to the lower edge. The field will resemble that sketched in Figure 4/5/2. Displacement from the origin O to A is followed by a return of the representative point to O, and this return corresponds to the stability. Displacement from O to B is followed by a departure from the region under consideration, and this departure corresponds to the instability. The uncertainty of the movements near O
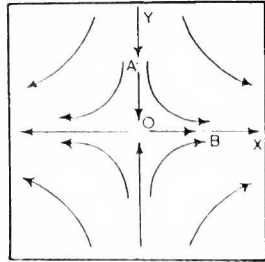
FIGURE 4/5/2.

corresponds to the uncertainty in the behaviour of the card when released from the vertical position.

The Watt's governor has a more complicated field, but an approximation may be obtained without difficulty. The system may be specified to an approximation sufficient for our purpose by three variables :

(x) the speed of the engine and governor (r.p.m.),

(y) the distance between the weights, or the position of the throttle, and

(z) the velocity of flow of the steam.

(y represents either of two quantities because they are rigidly connected). If, now, a disturbance suddenly accelerates the engine, increasing x, the increase in x will increase y ; this increase in y will be followed by a decrease of z, and then by a decrease of x. As the changes occur not in jumps but continuously, the line of behaviour must resemble that
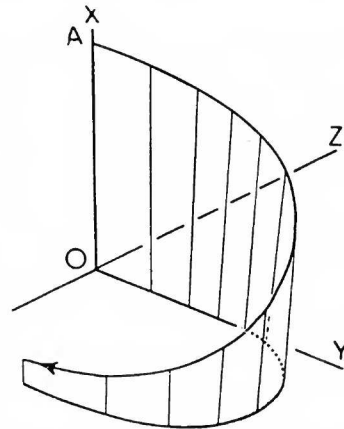

FIGURE 4/5/3 : One line of behaviour in the field of the Watt's governor. For clarity, the resting state of the system has been used as origin. The system has been displaced to A and then released,

sketched in Figure 4/5/3. The other lines of the field could be added by considering what would happen after other disturbances (lines starting from points other than A). Although having different initial states, all the lines would converge towards O.

4/6. In some of our examples, for instance that of the cube, the lines of behaviour terminate in a point at which all movement ceases. In other examples the movement does not wholly cease; many a thermostat settles down, when close to its resting state, to a regular small oscillation. We shall be little interested in the details of what happens at the exact centre.

4/7. More important is the underlying theme that in all cases the stable system is characterised by the fact that after a displacement we can assign some *limit* to the subsequent movement of the representative point, whereas in the unstable system such limitation is either impossible or depends on facts outside the subject of discussion. Thus, if a thermostat is set at $37°$ C. and displaced to $40°$, we can predict that in the future it will not go outside specified limits, which might be in one apparatus $36°$ and $40°$. On the other hand, if the thermostat has been assembled with a component reversed so that it is unstable (S. 4/12) and if it is displaced to $40°$, then we can give no limits to its subsequent temperatures; unless we introduce such new topics as the melting-point of its solder.

4/8. These considerations bring us to the definition which will be used. Given an absolute system and a region within its field, *a line of behaviour from a point within the region is* **stable** *if it never leaves the region*. Within one absolute system a change of the region or of the line of behaviour may change the result of the criterion.

Thus, in Figure 4/3/1 the stability around A can be decided thus : make a mark on each side of A so as to define the region ; then as the line of behaviour from any point within this region never leaves it, the line of behaviour is stable. On the other hand, no region can be found around B which gives a stable line of behaviour. Again, consider Figure 4/5/2 : a boundary line is first drawn to enclose A, O and B, in order to define which part of the field is being discussed. The line of behaviour from

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A is then found to be stable, and the line from *B* unstable. This example makes it obvious that the concept of ' stability ' belongs primarily to a line of behaviour, not to a whole field. In particular it should be noted that in all cases the definition gives a unique answer once the line, the region, and the initial state are given.

The examples above have been selected to test the definition severely. Sometimes the fields are simpler. In the field of the cube, for instance, it is possible to draw many boundaries, each oval in shape, such that all lines within the boundary are stable. The field of the Watt's governor is also of this type. It will be noticed that before we can discuss stability in a particular case we must always define which region of the phase-space we are referring to.

A field within a given region is ' stable ' if every line of behaviour in the region is stable. A system is ' stable ' if its field is stable.

4/9.   A resting state *is one from which an absolute system does not move when released.* Such states occur in Figure 4/3/1 at *A* and *B*, and in Figure 4/5/1 at the origin.

Although the variables do not change value when at a resting state this invariance does not imply that the ' machine ' itself is inactive. Thus, a steady Watt's governor implies that the engine is working at a non-zero rate. And a living muscle, even if unchanging in tension, is continually active in metabolism. ' Resting ' applies to the variables, not necessarily to the ' machine ' that yields the variables.
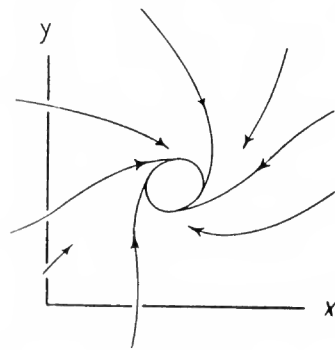

FIGURE 4/10/1.

4/10. If a line of behaviour is re-entrant to itself, the system undergoes a recurrent cycle. If the cycle is wholly contained in a given region, and the lines of behaviour lead *into* the cycle, the cycle is stable.

Such a cycle is commonly shown by thermostats which, after correcting any gross displacement, settle down to a steady oscillation. In such a case the field will show, not convergence to a point but convergence to a cycle, such as is shown exaggerated in Figure 4/10/1.

4/11. This definition of stability conforms to the requirement of S. 2/8; for the observed behaviour of the system determines the field, and the field determines the stability.

Feedback

4/12. The description given in S. 4/1 of the working of the Watt's governor showed that it is arranged in a functional circuit: the chain of cause and effect is re-entrant. Thus if we represent 'A has a direct effect on B' or 'A directly disturbs B' by the symbol $A \rightarrow B$, then the construction of the Watt's governor may be represented by the diagram :
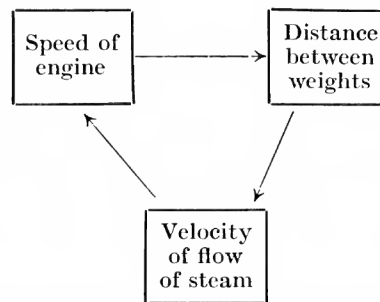

(The number of variables named here is partly optional.)

Lest the diagram should seem based on some metaphysical knowledge of causes and effects, its derivation from the actual machine, using only primary operations, will be described.

Suppose the relation between 'speed of engine' and 'distance between weights' is first investigated. The experimenter would fix the variable 'velocity of flow of steam'. Then he would try various speeds of the engine, and would observe how these changes affected the behaviour of 'distance between the weights'. He would find that changes in the speed of the engine were regularly followed by changes in the distance between the weights. He need know nothing of the nature of the ultimate physical linkages, but he would observe the fact. Then, still keeping 'velocity of flow of steam' constant, he would try various distances between the weights, and would observe the effect of such changes on the speed of the engine ; he would find them to be without effect.

He would thus have established that there is an arrow from left to right but not from right to left in
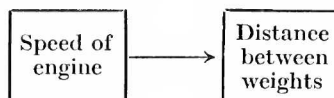
This procedure could then be applied to the two variables 'distance between weights' and 'velocity of flow of steam', while the other variable 'speed of engine' was kept constant. And finally the relations between the third pair could be established.
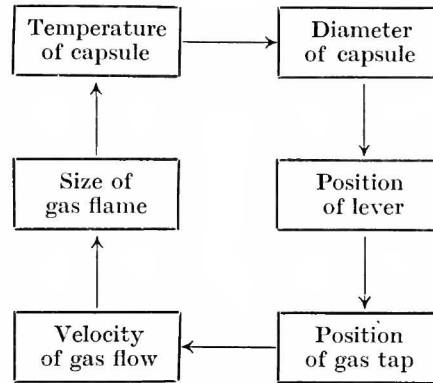
The method is clearly general. To find the immediate effects in a system with variables A, B, C, D . . . take one pair, A and B say; hold all other variables C, D . . . constant; note B's behaviour when A starts, or is held, at A_1; and also its behaviour when A starts, or is held, at A_2. If these behaviours of B are the same, then there is no immediate effect from A to B. But if the B's behaviours are unequal, and regularly depend on what value A starts from, or is held at, then there is an immediate effect, which we symbolise by $A \rightarrow B$.

By interchanging A and B in the process we can test for $B \rightarrow A$. And by using other pairs in turn we can determine all the immediate effects. The process is clearly defined, and consists purely of primary operations. It therefore uses no borrowed knowledge. We shall frequently use this **diagram of immediate effects.**

If A has an immediate effect on B, and B has an immediate effect on A, the relation will be represented by $A \rightleftarrows B$. If A affects B, and B also affects C, but A does not affect C directly, the relation will be shown by $A \rightarrow B \rightarrow C$. If there is a sequence of arrows joined head to tail and we are not interested in the intermediate steps, the sequence may often be contracted without ambiguity to $A \rightarrow C$. The diagram will be used only for illustration and not for rigorous proofs, so further precision is not required. (It should be carefully distinguished from the diagram of ' ultimate ' effects, but this is not required yet and will be described in S. 14/6. At the moment we regard the concept of one variable 'having an effect' on another as well understood. But the concept will be examined more closely, and given more precision, in S. 14/3.)
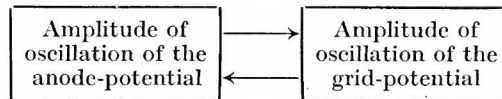
A gas thermostat also shows a functional circuit or feedback;

for if it is controlled by a capsule which by its swelling moves a lever which controls the flow of gas to the heating flame, the diagram of immediate effects would be :

```
┌──────────────┐        ┌──────────────┐
│ Temperature  │───────▶│  Diameter    │
│  of capsule  │        │  of capsule  │
└──────────────┘        └──────────────┘
       ▲                        │
       │                        ▼
┌──────────────┐        ┌──────────────┐
│   Size of    │        │  Position    │
│  gas flame   │        │  of lever    │
└──────────────┘        └──────────────┘
       ▲                        │
       │                        ▼
┌──────────────┐        ┌──────────────┐
│  Velocity    │◀───────│  Position    │
│ of gas flow  │        │  of gas tap  │
└──────────────┘        └──────────────┘
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The reader should verify that each arrow represents a physical action which can be demonstrated if all variables other than the pair are kept constant.

Another example is provided by ' reaction ' in a radio receiver. We can represent the action by two variables linked in two ways :

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┌──────────────┐        ┌──────────────┐
│ Amplitude of │───────▶│ Amplitude of │
│oscillation of│        │oscillation of│
│ the anode-   │◀───────│ the grid-    │
│  potential   │        │  potential   │
└──────────────┘        └──────────────┘
```

The lower arrow represents the grid-potential's effect within the valve on the anode-current. The upper arrow represents some arrangement of the circuit by which fluctuation in the anode-potential affects the grid-potential. The effect represented by the lower arrow is determined by the valve-designer, that of the upper by the circuit-designer.

Such systems whose variables affect one another in a circuit possess what the radio-engineer calls ' feedback ' ; they are also sometimes described as ' servo-mechanisms '. They are at least as old as the Watt's governor and may be older. But only during the last decade has it been realised that the possession of feedback gives a machine potentialities that are not available to a machine lacking it. The development occurred mainly during the last war, stimulated by the demand for automatic methods of control

of searchlight, anti-aircraft guns, rockets, and torpedoes, and facilitated by the great advances that had occurred in electronics. As a result, a host of new machines appeared which acted with powers of self-adjustment and correction never before achieved. Some of their main properties will be described in S. 4/14.

The nature, degree, and polarity of the feedback has a decisive effect on the stability or instability of the system. In the Watt's governor or in the thermostat, for instance, the connection of a part in reversed position, reversing the polarity of action of one component on the next, may, and probably will, turn the system from stable to unstable. In the reaction circuit of the radio set, the stability or instability is determined by the quantitative relation between the two effects.

Instability in such systems is shown by the development of a 'runaway'. The least disturbance is magnified by its passage round the circuit so that it is incessantly built up into a larger
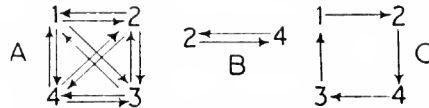

FIGURE 4/12/1.

and larger deviation from the resting state. The phenomenon is identical with that referred to as a 'vicious circle'.

The examples shown have only a simple circuit. But more complex systems may have many interlacing circuits. If, for instance, as in S. 8/8, four variables all act on each other, the diagram of immediate effects would be that shown in Figure 4/12/1 (*A*). It is easy to verify that such a system contains twenty interlaced circuits, two of which are shown at *B* and *C*.

The further development of the theory of systems with feedback cannot be made without mathematics. But here it is sufficient to note two facts : a system which possesses feedback is usually actively stable or actively unstable ; and whether it is stable or unstable depends on the quantitative details of the particular arrangement.

4/13. It will be noticed that stability, as defined, in no way implies fixity or rigidity. It is true that stable systems may have a resting state at which they will show no change ; but the lack

of change is deceptive if it suggests rigidity : they have only to be disturbed to show that they are capable of extensive and active movements. They are restricted only in that they do not show the unlimited divergencies of instability.

Goal-seeking

4/14. Every stable system has the property that if displaced from a resting state and released, the subsequent movement is so matched to the initial displacement that the system is brought back to the resting state. A variety of disturbances will therefore evoke a variety of matched reactions. Reference to a simple field such as that of Figure 4/5/1 will establish the point.

This pairing of the line of return to the initial displacement has sometimes been regarded as ' intelligent ' and peculiar to living things. But a simple refutation is given by the ordinary pendulum : if we displace it to the right, it develops a force which tends to move it to the left ; and if we displace it to the left, it develops a force which tends to move it to the right. Noticing
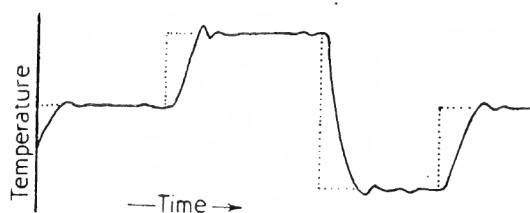

FIGURE 4/14/1 : Tracing of the temperature (solid line), of a thermostatically controlled bath, and of the control setting (broken line).

that the pendulum reacted with forces which though varied in direction always pointed towards the centre, the mediaeval scientist would have said ' the pendulum seeks the centre '. By this phrase he would have recognised that the behaviour of a stable system may be described as ' goal-seeking '. Without introducing any metaphysical implications we may recognise that this type of behaviour does occur in the stable dynamic systems. Thus Figure 4/14/1 shows how, as the control setting of a thermostat was altered, the temperature of the apparatus always followed it, the set temperature being treated as if it were a goal.

Such a movement occurs here in only one dimension (tempera-

ture), but other goal-seeking devices may use more. The radar-controlled searchlight, for example, uses the reflected impulses to alter its direction of aim so as to minimise the angle between its direction of aim and the bearing of the source of the reflected impulses. So if the aircraft swerves, the searchlight will follow it actively, just as the temperature followed the setting.

The examples show the common feature that each is ' error-controlled ' : each is partly controlled by the deviation of the system's state from the resting state (which, in these examples, can be moved by an outside operation). The thermostat is affected by the difference between the actual and the set temperatures. The searchlight is affected by the difference between the two directions. So it will be seen that machines with feedback are not subject to the oft-repeated dictum that machines must act blindly and cannot correct their errors. Such a statement is true of machines without feedback, but not of machines in general.

Once it is appreciated that feedback can be used to correct any deviation we like, it is easy to understand that there is no limit to the complexity of goal-seeking behaviour which may occur in machines quite devoid of any ' vital ' or ' intelligent ' factor. Thus, an automatic anti-aircraft gun may be controlled by the radar-pulses reflected back both from the target aeroplane and from its own bursting shells, in such a way that it tends to minimise the distance between shell-burst and plane. Such a system, wholly automatic, cannot be distinguished by its behaviour from a humanly operated gun : both will fire at the target, following it through all manoeuvres, continually using the errors to improve the next shot. It will be seen, therefore, that a system with feedback may be both wholly automatic and yet actively and complexly goal-seeking. There is no incompatibility.

4/15. An important feature of a system's stability (or instability) is that it is a property of the whole system and can be assigned to no part of it. The statement may be illustrated by a consideration of the third diagram of S. 4/12 as it is related to the practical construction of the thermostat. In order to ensure the stability of the final assembly, the designer must consider :

(1) The effect of the temperature on the diameter of the capsule, i.e. whether a rise in temperature makes the capsule expand or shrink.

(2) Which way an expansion of the capsule moves the lever.

(3) Which way a movement of the lever moves the gas-tap.

(4) Whether a given movement of the gas-tap makes the velocity of gas-flow increase or decrease.

(5) Whether an increase of gas-flow makes the size of the gas-flame increase or decrease.

(6) How an increase in size of the gas-flame will affect the temperature of the capsule.

Some of the answers are obvious, but they must none the less be included. When the six answers are known, the designer can ensure stability only by arranging the components (chiefly by manipulating (2), (3) and (5)) so that as a whole they form an appropriate combination. Thus five of the effects may be decided, yet the stability will still depend on how the sixth is related to them. The stability belongs only to the combination; it cannot be related to the parts considered separately.

In order to emphasise that the stability of a system is independent of any conditions which may hold over the parts which compose the whole, some further examples will be given. (Proofs of the statements will be found in S. 21/5–7.)

(*a*) Two systems may be joined so that they act and interact on one another to form a single system : to know that the two systems when separate were both stable is to know nothing about the stability of the system formed by their junction : it may be stable or unstable.

(*b*) Two systems, both unstable, may join to form a whole which is stable.

(*c*) Two systems may form a stable whole if joined in one way, and may form an unstable whole if joined in another way.

(*d*) In a stable system the effect of fixing a variable may be to render the remainder unstable.

Such examples could be multiplied almost indefinitely. They illustrate the rule that the stability (or instability) of a dynamic system depends on the parts and their interrelations as a whole.

4/16. The fact that the stability of a system is a property of the system as a whole is related to the fact that the presence of stability (as contrasted with instability) always implies some co-ordination of the actions between the parts. In the thermostat the necessity for co-ordination is clear, for if the components were assembled

at random there would be only an even chance that the assembly would be stable. But as the system and the feedbacks become more complex, so does the achievement of stability become more difficult and the likelihood of instability greater. Radio engineers know only too well how readily complex systems with feedback become unstable, and how difficult is the discovery of just that combination of parts and linkages which will give stability.

The subject is discussed more fully in S. 20/12 : here it is sufficient to note that as the number of variables increases so usually do the effects of variable on variable have to be co-ordinated with more and more care if stability is to be achieved.

REFERENCE

WIENER, NORBERT. *Cybernetics.* New York, 1948.