

CHAPTER 21

Parameters

21/1. WITH canonical equations

$$\frac{dx_i}{dt} = f_i(x_1, \dots, x_n) \quad (i = 1, \dots, n),$$

the form of the field is determined by the functional forms f_i regarded as functions of x_1, \dots, x_n . If parameters a_1, a_2, \dots are taken into consideration, the system will be specified by equations

$$\frac{dx_i}{dt} = f_i(x_1, \dots, x_n; a_1, a_2, \dots) \quad (i = 1, \dots, n).$$

If the parameters are constant, the x 's continue to form an absolute system. If the a 's can take m combinations of values, then the x 's form m different absolute systems, and will show m different fields. If a parameter can change continuously (in value, not in time), no limit can be put to the number of different fields which can arise.

If a parameter affects only certain variables directly, it will appear only in the corresponding f 's. Thus, if it affects only x_1 directly, so that the diagram of immediate effects is

$$a \rightarrow x_1 \rightleftarrows x_2,$$

then a will appear only in f_1 :

$$\begin{aligned} dx_1/dt &= f_1(x_1, x_2; a) \\ dx_2/dt &= f_2(x_1, x_2). \end{aligned}$$

But it will in general appear in all the F 's of the integrals (S. 19/10). The subject is developed further in Chapter 24.

Change of parameters can represent *every* alteration which can be made on an absolute system, and therefore on any physical or biological 'machine'. It includes every possibility of experimental interference. Thus if a set of variables that are joined to form the system $\dot{x} = f(x)$ are changed in their relations so that they form the system $\dot{x} = \phi(x)$, then the change can equally

well be represented as a change in the single system $\dot{x} = \psi(x; \alpha)$. For if α can take two values, 1 and 2 say, and if

$$\begin{aligned} f(x) &\equiv \psi(x; 1) \\ \phi(x) &\equiv \psi(x; 2) \end{aligned}$$

then the two representations are identical.

As example of its method, the action of S. 8/10, where the two front magnets of the homeostat were joined by a light glass fibre and so forced to move from side to side together, will be shown so that the joining and releasing are equivalent in the canonical equations to a single parameter taking one of two values.

Suppose that units x_1 , x_2 and x_3 were used, and that the magnets of 1 and 2 were joined. Before joining, the equations were (S. 19/11)

$$\left. \begin{aligned} dx_1/dt &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ dx_2/dt &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ dx_3/dt &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \right\}$$

After joining, x_2 can be ignored as a variable since x_1 and x_2 are effectively only a single variable. But x_2 's output still affects the others, and its force still acts on the fibre. The equations therefore become

$$\left. \begin{aligned} dx_1/dt &= (a_{11} + a_{12} + a_{21} + a_{22})x_1 + (a_{13} + a_{23})x_3 \\ dx_3/dt &= (a_{31} + a_{32})x_1 + a_{33}x_3 \end{aligned} \right\}$$

It is easy to verify that if the full equations, including the parameter b , were :

$$\left. \begin{aligned} dx_1/dt &= \{a_{11} + b(a_{12} + a_{21} + a_{22})\}x_1 + (1 - b)a_{12}x_2 \\ &\quad + (a_{13} + ba_{23})x_3 \\ dx_2/dt &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ dx_3/dt &= (a_{31} + ba_{32})x_1 + (1 - b)a_{32}x_2 + a_{33}x_3 \end{aligned} \right\}$$

then the joining and releasing are identical in their effects with giving b the values 1 and 0 respectively. (These equations are sufficient but not, of course, necessary.)

21/2. A variable x_k behaves as a 'null-function' if it has the following properties, which are easily shown to be necessary and sufficient for each other :

- (1) As a function of the time, it remains at its initial value x_k^0 .
- (2) In the canonical equations, $f_k(x_1, \dots, x_n)$ is identically zero.

(3) In the group equations, $F_k(x_1^0, \dots, x_n^0; t) \equiv x_k^0$.

(Some region of the phase-space is assumed given.)

Since we usually consider absolute systems, we shall usually require the parameters to be held constant. Since null-functions also remain constant, the properties of the two will often be similar. (A fundamental distinction by definition is that parameters are outside, while null-functions may be inside, the given system.)

21/3. In an absolute system, the variables other than the step- and null-functions will be referred to as **main** variables.

21/4. Theorem : In an absolute system, the system of the main-variables forms an absolute subsystem provided no step-function changes from its initial value.

Suppose x_1, \dots, x_k are null- and step-functions and the main-variables are x_{k+1}, \dots, x_n . The canonical equations of the whole system are

$$\left. \begin{aligned} dx_1/dt &= 0 \\ &\dots\dots\dots \\ dx_k/dt &= 0 \\ dx_{k+1}/dt &= f_{k+1}(x_1, \dots, x_k, x_{k+1}, \dots, x_n) \\ &\dots\dots\dots \\ dx_n/dt &= f_n(x_1, \dots, x_k, x_{k+1}, \dots, x_n) \end{aligned} \right\}$$

The first k equations can be integrated at once to give $x_1 = x_1^0, \dots, x_k = x_k^0$. Substituting these in the remaining equations we get :

$$\left. \begin{aligned} dx_{k+1}/dt &= f_{k+1}(x_1^0, \dots, x_k^0, x_{k+1}, \dots, x_n) \\ &\dots\dots\dots \\ dx_n/dt &= f_n(x_1^0, \dots, x_k^0, x_{k+1}, \dots, x_n) \end{aligned} \right\}$$

The terms x_1^0, \dots, x_k^0 are now constants, not effectively functions of t at all. The equations are in canonical form, so the system is absolute over any interval not containing a change in x_1^0, \dots, x_k^0 .

Usually the selection of variables to form an absolute system is rigorously determined by the real, natural relationships existing in the real 'machine', and the observer has no power to alter them without making alterations in the 'machine' itself. The theorem, however, shows that without affecting the absoluteness we may take

null-functions into the system or remove them from it as we please.

It also follows that the statements: 'parameter a was held constant at a^0 ', and 'the system was re-defined to include a , which, as a null-function, remained at its initial value of a^0 ' are merely two ways of describing the same facts.

21/5. The fact that the field is changed by a change of parameter implies that the stabilities of the lines of behaviour are changed. For instance, consider the system

$$dx/dt = -x + ay, \quad dy/dt = x - y + 1$$

where x and y have been used for simplicity instead of x_1 and x_2 . When $a = 0, 1$, and 2 respectively, the system has the three fields shown in Figure 21/5/1.

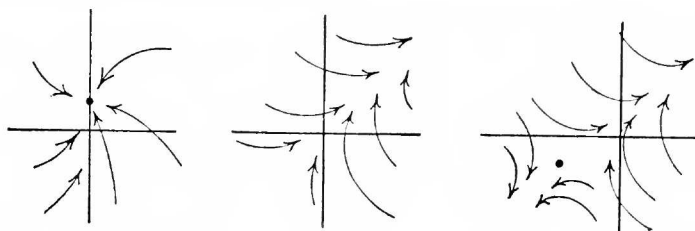


FIGURE 21/5/1 : Three fields of x and y when a has the values (left to right) $0, 1$, and 2 .

When $a = 0$ there is a stable resting state at $x = 0, y = 1$;

when $a = 1$ there is no resting state;

when $a = 2$ there is an unstable resting state at $x = -2, y = -1$.

The system has as many fields as there are values to a .

21/6. The simple physical act of joining two machines has, of course, a counterpart in the equations, shown more simply in the canonical than in the group equations.

One could, of course, simply write down equations in all the variables and then simply let some parameter a have one value when the parts are joined and another when they are separated. This method, however, gives no insight into the real events in 'joining' two systems. A better method is to equate parameters in one system to variables in the other. When this is

done, the second dominates the first. If parameters in each are equated to variables in the other, then a two-way interaction occurs. For instance, suppose we start with the 2-variable system

$$\left. \begin{aligned} dx/dt &= f_1(x, y; a) \\ dy/dt &= f_2(x, y) \end{aligned} \right\} \text{and the 1-variable system } dz/dt = \phi(z; b)$$

then the diagram of immediate effects is

$$a \rightarrow x \rightleftarrows y \quad b \rightarrow z$$

If we put $a = z$, the new system has the equations

$$\left. \begin{aligned} dx/dt &= f_1(x, y; z) \\ dy/dt &= f_2(x, y) \\ dz/dt &= \phi(z; b) \end{aligned} \right\}$$

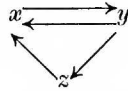
and the diagram of immediate effects becomes

$$b \rightarrow z \rightarrow x \rightleftarrows y.$$

If a further join is made by putting $b = y$, the equations become

$$\left. \begin{aligned} dx/dt &= f_1(x, y; z) \\ dy/dt &= f_2(x, y) \\ dz/dt &= \phi(z; y) \end{aligned} \right\}$$

and the diagram of immediate effects becomes



In this method each linkage uses up one parameter. This is reasonable; for the parameter used by the other system might have been used by the experimenter for arbitrary control. So the method simply exchanges the experimenter for another system.

This method of joining does no violence to each system's internal activities: these proceed as before except as modified by the actions coming in through the variables which were once parameters.

21/7. The stabilities of separate systems do not define the stability of the system formed by joining them together.

In the general case, when the f 's are unrestricted, this proposition is not easily given a meaning. But in the linear case (to

which all continuous systems approximate, S. 20/4) the meaning is clear. Several examples will be given.

Example 1: Two systems may be stable if joined one way, and unstable if joined another. Consider the 1-variable systems $dx/dt = x + 2p_1 + p_2$ and $dy/dt = -2r - 3y$. If they are joined by putting $r = x$, $p_1 = y$, the system becomes

$$\left. \begin{aligned} dx/dt &= x + 2y + p_2 \\ dy/dt &= -2x - 3y \end{aligned} \right\}$$

The latent roots of its matrix are $-1, -1$; so it is stable. But if they are joined by $r = x$, $p_2 = y$, the roots become $+0.414$ and -2.414 ; and it is unstable.

Example 2: Several systems, all stable, may be unstable when joined. Join the three systems

$$\left. \begin{aligned} dx/dt &= -x - 2q - 2r \\ dy/dt &= -2p - y + r \\ dz/dt &= p + q - z \end{aligned} \right\}$$

all of which are stable, by putting $p = x$, $q = y$, $r = z$. The resulting system has latent roots $+1, -2, -2$.

Example 3: Systems, each unstable, may be joined to form a stable whole. Join the 2-variable system

$$\left. \begin{aligned} dx/dt &= 3x - 3y - 3p \\ dy/dt &= 3x - 9y - 8p \end{aligned} \right\}$$

which is unstable, to $dz/dt = 21q + 3r + 3z$, which is also unstable, by putting $q = x$, $r = y$, $p = z$. The whole is stable.

Example 4: If a system

$$dx_i/dt = f_i(x_1, \dots, x_n; a_1, \dots) \quad (i = 1, \dots, n)$$

is joined to another system, of y 's, by equating various a 's and y 's, then the resting states that were once given by certain combinations of x and a will still occur, so far as the x -system is concerned, when the y 's take the values the a 's had before. The zeros of the f 's are thus invariant for the operations of joining and separating.