CHAPTER 20

Stability

20/1. 'STABILITY' is defined primarily as a relation between a line of behaviour and a region in phase-space because only in this way can we get a test that is unambiguous in all possible cases. Given an absolute system and a region within its field, a line of behaviour from a point within the region is **stable** if it never leaves the region.

20/2. If all the lines within a given region are stable from all points within the region, and if all the lines meet at one point, the system has 'normal' stability.

20/3. A **resting state** can be defined in several ways. In the field it is a terminating point of a line of behaviour. In the group equations of S. 19/10 the resting state $X_1, \ldots, X_n$ is given by the equations

$$X_i = \lim_{t \to \infty} F_i(x^0;\ t) \qquad (i = 1, \ldots, n) \qquad . \quad (1)$$

if the n limits exist. In the canonical equations the values satisfy

$$f_i(X_1, \ldots, X_n) = 0 \qquad (i = 1, \ldots, n) \qquad . \quad (2)$$

A resting state is an invariant of the group, for a change of t does not alter its value.

If the Jacobian of the f's, i.e. the determinant $\left|\frac{\partial f_i}{\partial x_j}\right|$, which will be symbolised by J, is not identically zero, then there will be isolated resting states. If $J \equiv 0$, but not all its first minors are zero, then the equations define a curve, every point of which is a resting state. If $J \equiv 0$ and all first minors but not all second minors are zero, then a two-way surface exists composed of resting states; and so on.

20/4. *Theorem:* If the f's are continuous and differentiable, an absolute system tends to the linear form (S. 19/27) in the neighbourhood of a resting state.

Let the system, specified by

$$dx_i/dt = f_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n)$$

have a resting state $X_1, \ldots, X_n$, so that

$$f_i(X_1, \ldots, X_n) = 0 \qquad (i = 1, \ldots, n)$$

Put $x_i = X_i + \xi_i$ $(i = 1, \ldots, n)$ so that x_i is measured as a deviation ξ_i from its resting value. Then

$$\frac{d}{dt}(X_i + \xi_i) = f_i(X_1 + \xi_1, \ldots, X_n + \xi_n) \qquad (i = 1, \ldots, n)$$

Expanding the right-hand side by Taylor's theorem, noting that $dX_i/dt = 0$ and that $f_i(X) = 0$, we find, if the ξ's are infinitesimal, that

$$\frac{d\xi_i}{dt} = \frac{\partial f_i}{\partial \xi_1}\xi_1 + \ldots + \frac{\partial f_i}{\partial \xi_n}\xi_n \qquad (i = 1, \ldots, n)$$

The partial derivatives, taken at the point $X_1, \ldots, X_n$, are numerical constants. So the system is linear.

20/5. In general the only test for stability is to observe or compute the given line of behaviour and to see what happens as $t \to \infty$. For the linear system, however, there are tests that do not involve the line of behaviour explicitly. Since, by the previous section, many systems approximate to the linear within the region in which we are interested, the methods to be described are widely applicable.

Let the linear system be

$$\frac{dx_i}{dt} = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \qquad (i = 1, \ldots, n) \quad (1)$$

or, in the concise matrix notation (S. 19/27)

$$\dot{x} = Ax \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Constant terms on the right-hand side make no difference to the stability and can be ignored. If the determinant of A is not zero, there is a single resting state. The determinant

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix}$$

when expanded gives a polynomial in λ of degree n which, when equated to 0, gives the **characteristic equation** of the matrix A:

$$\lambda^n + m_1\lambda^{n-1} + m_2\lambda^{n-2} + \ldots + m_n = 0.$$

20/6. Each coefficient m_i is the sum of all i-rowed principal (co-axial) minors of A, multiplied by $(-1)^i$. Thus,

$$m_1 = -(a_{11} + a_{22} + \ldots + a_{nn})\,; \quad m_n = (-1)^n \,|A|.$$

Example : The linear system

$$\left.\begin{aligned} dx_1/dt &= -5x_1 + 4x_2 - 6x_3 \\ dx_2/dt &= 7x_1 - 6x_2 + 8x_3 \\ dx_3/dt &= -2x_1 + 4x_2 - 4x_3 \end{aligned}\right\}$$

has the characteristic equation

$$\lambda^3 + 15\lambda^2 + 2\lambda + 8 = 0.$$

20/7. Of this equation the roots $\lambda_1, \ldots, \lambda_n$ are the **latent roots** of A. The integral of the canonical equations gives each x_i as a linear function of the exponentials $e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$. For the sum to be convergent, no real part of $\lambda_1, \ldots, \lambda_n$ must be positive, and this criterion provides a test for the stability of the system.

Example : The equation $\lambda^3 + 15\lambda^2 + 2\lambda + 8 = 0$ has roots $-14{\cdot}902$ and $-0{\cdot}049 \pm 0{\cdot}729\sqrt{-1}$, so the system of the previous section is stable.

20/8. A test which avoids finding the latent roots is Hurwitz' : a necessary and sufficient condition that the linear system is stable is that the series of determinants

$$m_1, \quad \begin{vmatrix} m_1 & 1 \\ m_3 & m_2 \end{vmatrix}, \quad \begin{vmatrix} m_1 & 1 & 0 \\ m_3 & m_2 & m_1 \\ m_5 & m_4 & m_3 \end{vmatrix}, \quad \begin{vmatrix} m_1 & 1 & 0 & 0 \\ m_3 & m_2 & m_1 & 1 \\ m_5 & m_4 & m_3 & m_2 \\ m_7 & m_6 & m_5 & m_4 \end{vmatrix}, \text{ etc.}$$

(where, if $q > n$, $m_q = 0$), are all positive.

Example : The system with characteristic equation

$$\lambda^3 + 15\lambda^2 + 2\lambda + 8 = 0$$

yields the series

$$+15, \quad \begin{vmatrix} 15 & 1 \\ 8 & 2 \end{vmatrix}, \quad \begin{vmatrix} 15 & 1 & 0 \\ 8 & 2 & 15 \\ 0 & 0 & 8 \end{vmatrix}.$$

These have the values $+15$, $+22$, and $+176$. So the system is stable, agreeing with the previous test.

20/9. If the coefficients in the characteristic equation are not all positive the system is unstable. But the converse is not true. Thus the linear system whose matrix is

$$\begin{bmatrix} 1 & \sqrt{6} & 0 \\ -\sqrt{6} & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

has the characteristic equation $\lambda^3 + \lambda^2 + \lambda + 21 = 0$; but the latent roots are $+1 \pm \sqrt{-6}$ and -3; so the system is unstable.

20/10. Another test, related to Nyquist's, states that a linear system is stable if, and only if, the polynomial

$$\lambda^n + m_1\lambda^{n-1} + m_2\lambda^{n-2} + \ldots + m_n$$

changes in amplitude by $n\pi$ when λ, a complex variable ($\lambda = a + b\iota$ where $\iota = \sqrt{-1}$), goes from $-\iota\,\infty$ to $+\iota\,\infty$ along the b-axis in the complex λ-plane.

Nyquist's criterion of stability is widely used in the theory of electric circuits and of servo-mechanisms. It, however, uses data obtained from the response of the system to persistent harmonic disturbance. Such disturbance renders the system non-absolute and is therefore based on an approach different from ours.

20/11. Some further examples will illustrate various facts relating to stability.

Example 1: If a matrix $[a]$ of order $n \times n$ has latent roots $\lambda_1, \ldots, \lambda_n$, then the matrix, written in partitioned form,

$$\left[\begin{array}{c|c} 0 & I \\ \hline a & 0 \end{array}\right]$$

of order $2n \times 2n$, where I is the unit matrix, has latent roots $\pm\sqrt{\lambda_1}, \ldots, \pm\sqrt{\lambda_n}$. It follows that the system

$$\frac{d^2x_i}{dt^2} = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \qquad (i = 1, \ldots, n)$$

of common physical occurrence, must be unstable.

Example 2: The diagonal terms a_{ii} represent the intrinsic stabilities of the variables; for if all variables other than x_i are held constant, the linear system's i-th equation becomes

$$dx_i/dt = a_{ii}x_i + c,$$

where c is a constant, showing that under these conditions x_i will converge to $-c/a_{ii}$ if a_{ii} be negative, and will diverge without limit if a_{ii} be positive.

If the diagonal terms a_{ii} are much larger in absolute magnitude than the others, the roots tend to the values of a_{ii}. It follows that if the diagonal terms take extreme values they determine the stability.

Example 3: If the terms a_{ij} in the first $n-1$ rows (or columns) are given, the remaining n terms can be adjusted to make the latent roots take any assigned values.

Example 4: The matrix of the homeostat equations of S. 19/11 is

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ a_{11}h & a_{12}h & a_{13}h & a_{14}h & -j & \cdot & \cdot & \cdot \\ a_{21}h & a_{22}h & a_{23}h & a_{24}h & \cdot & -j & \cdot & \cdot \\ a_{31}h & a_{32}h & a_{33}h & a_{34}h & \cdot & \cdot & -j & \cdot \\ a_{41}h & a_{42}h & a_{43}h & a_{44}h & \cdot & \cdot & \cdot & -j \end{bmatrix}$$

If $j = 0$, the system must be unstable (by Example 1 above). If the matrix has latent roots $\mu_1, \ldots, \mu_8$, and if $\lambda_1, \ldots, \lambda_4$ are the latent roots of the matrix $[a_{ij}h]$, and if $j \neq 0$, then the λ's and μ's are related by $\lambda_p = \mu_q^2 + j\mu_q$. As $j \to \infty$ the 8-variable and the 4-variable systems are stable or unstable together.

Example 5: In a stable system, fixing a variable may make the system of the remainder unstable. For instance, the system with matrix

$$\begin{bmatrix} 6 & 5 & -10 \\ -4 & -3 & -1 \\ 4 & 2 & -6 \end{bmatrix}$$

is stable. But if the third variable is fixed, the system of the first two variables has matrix

$$\begin{bmatrix} 6 & 5 \\ -4 & -3 \end{bmatrix}$$

and is unstable.

Example 6: Making one variable more stable intrinsically

(Example 2 of this section) may make the whole unstable. For instance, the system with matrix

$$\begin{bmatrix} -4 & -3 \\ 3 & 2 \end{bmatrix}$$

is stable. But if a_{11} becomes more negative, the system becomes unstable when a_{11} becomes more negative than $-4\frac{1}{2}$.

Example 7: In the $n \times n$ matrix

$$\left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right]$$

in partitioned form, $[a]$ is of order $k \times k$. If the k diagonal elements a_{ii} become much larger in absolute value than the rest, the latent roots of the matrix tend to the k values a_{ii} and the $n - k$ latent roots of $[d]$. Thus the matrix, corresponding to $[d]$,

$$\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

has latent roots $+1{\cdot}5 \pm 1{\cdot}658\iota$, and the matrix

$$\begin{bmatrix} -100 & -1 & 2 & 0 \\ -2 & -100 & -1 & 2 \\ 0 & -3 & 1 & -3 \\ 2 & -1 & 1 & 2 \end{bmatrix}$$

has latent roots $-101{\cdot}39$, $-98{\cdot}62$, and $+1{\cdot}506 \pm 1{\cdot}720\iota$.

Corollary: If system $[d]$ is unstable but the whole 4-variable system is stable, then making x_1 and x_2 more stable intrinsically will eventually make the whole unstable.

Example 8: The holistic nature of stability is well shown by the system with matrix

$$\begin{bmatrix} -3 & -2 & 2 \\ -6 & -5 & 6 \\ -5 & 2 & -4 \end{bmatrix}$$

in which each variable individually, and every pair, is stable; yet the whole is unstable.

The probability of stability

20/12. The probability that a system should be stable can be made precise by the point of view of S. 14/16. We consider an *ensemble* of absolute systems

$$dx_i/dt = f_i(x_1, \ldots, x_n; \ \alpha_1, \ldots) \qquad (i = 1, \ldots, n)$$

with parameters α_j, such that each combination of α-values gives an absolute system. We nominate a point Q in phase-space, and then define the 'probability of stability at Q' as the proportion of α-combinations (drawn as samples from known distributions) that give both (1) a resting state at Q, and (2) stable equilibrium at that point. The system's general 'probability of stability' is the probability at Q averaged over all Q-points. As the probability will usually be zero if Q is a point, we can consider instead the infinitesimal probability dp given when the point is increased to an infinitesimal volume dV.

The question is fundamental to our point of view; for, having decided that stability is necessary for homeostasis, we want to get a system of 10^{10} nerve-cells and a complex environment stable by some method that does not demand the improbable. The question cannot be treated adequately without some quantitative study. Unfortunately, the quantitative study involves mathematical difficulties of a high order. Non-linear systems cannot be treated generally but only individually. Here I shall deal only with the linear case. It is not implied that the nervous system is linear in its performance or that the answers found have any quantitative application to it. The position is simply that, knowing nothing of what to expect, we must collect what information we can so that we shall have at least some fixed points around which the argument can turn.

The applicability of the concept of linearity is considerably widened by the theorem of S. 20/4.

The problem may be stated as follows: A matrix of order $n \times n$ has elements which are real and are random samples from given distributions. Find the probability that all the latent roots have non-positive real parts.

This problem seems to be still unsolved even in the special cases in which all the elements have the same distributions, selected to be simple, as the 'normal' type e^{-x^2}, or the 'rectangular' type, constant between $-a$ and $+a$. Nevertheless, some answer is desirable, so the 'rectangular' distribution (integers evenly distributed between -9 and $+9$) was tested empirically. Matrices were formed from Fisher and Yates' Table of Random

Numbers, and each matrix was then tested for stability by Hurwitz' rule (S. 20/8 and S. 20/9). Thus a typical 3×3 matrix was

$$\begin{bmatrix} -1 & -3 & -8 \\ -5 & 4 & -2 \\ -4 & -4 & -9 \end{bmatrix}$$

In this case the second determinant is -86, so it need not be tested further as it is unstable by S. 20/9. The testing becomes very time-consuming when the matrices exceed 3×3, for the time taken increases approximately as n^5. The results are summarised in Table 20/12/1.

Order of matrix	Number tested	Number found stable	Per cent stable
2×2	320	77	24
3×3	100	12	12
4×4	100	1	1

TABLE 20/12/1.

The main feature is the rapidity with which the probability tends to zero. The figures given are compatible ($\chi^2 = 4{\cdot}53$, $P = 0{\cdot}10$) with the hypothesis that the probability for a matrix of order $n \times n$ is $1/2^n$. That this may be the correct expression for this particular case is suggested partly by the fact that it may be proved so when $n = 1$ and $n = 2$, and partly by the fact that, for stability, the matrix has to pass all of n tests. And in fact about a half of the matrices failed at each test. If the signs of the determinants in Hurwitz' test are statistically independent, then $1/2^n$ would be the probability.

In these tests, the intrinsic stabilities of the variables, as judged by the signs of the terms in the main diagonal, were equally likely to be stable or unstable. An interesting variation, therefore, is to consider the case where the variables are all intrinsically stable (all terms in the main diagonal distributed uniformly between 0 and -9).

The effect is to increase their probability of stability. Thus when n is 1 the probability is 1 (instead of $\frac{1}{2}$); and when n is

2 the probability is 3/4 (instead of 1/4). Some empirical tests gave the results of Table 20/12/2.

Order of matrix	Number tested	Number found stable	Per cent stable
2 × 2	120	87	72
3 × 3	100	55	55

TABLE 20/12/2.

The probability is higher, but it still falls as n is increased.

A similar series of tests was made with the homeostat. Units were allowed to interact with settings determined by the uni-selectors, and the percentage of stable combinations found when the number of units was two; the percentage was then found for the same general conditions except that three units interacted;

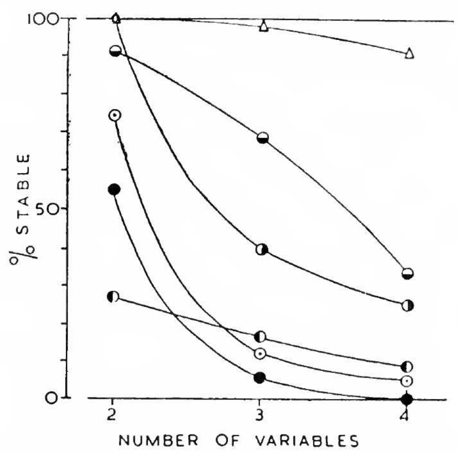

FIGURE 20/12/1.

and then four. The general conditions were then changed and a new triple of percentages found. And this was repeated six times altogether. As the general conditions sometimes encouraged, sometimes discouraged, stability, some of the triples were all high, some all low; but in every case the per cent stable fell as the number of interacting units was increased. The results are given in Figure 20/12/1.

These results prove little ; but they suggest that the probability of stability is small in large systems assembled at random. It is suggested, therefore, that large systems should be assumed unstable unless evidence to the contrary can be produced.

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