CHAPTER 19

The Absolute System

(Some of the definitions already given are repeated here for convenience)

19/1. A system of \( n \) variables will usually be represented by \( x_1, \ldots, x_n \), or sometimes more briefly by \( x \). \( n \) will be assumed finite; a system with an infinite number of variables (e.g. that of S. 19/23), where \( x_i \) is a continuous function of \( i \), will be replaced by a system in which \( i \) is discontinuous and \( n \) finite, and which differs from the original system by some negligible amount.

19/2. Each variable \( x_i \) is a function of the time \( t \); it will sometimes be written as \( x_i(t) \) for emphasis. It must be single-valued, but need not be continuous. A constant may be regarded as a variable which undergoes zero change.

19/3. The state of a system at a time \( t \) is the set of numerical values of \( x_1(t), \ldots, x_n(t) \). Two states are 'equal' if \( n \) equalities exist between the corresponding pairs.

19/4. A line of behaviour is specified by a succession of states and the time-intervals between them. Two lines of behaviour which differ only in the absolute times of their initial states are equal.

19/5. A geometrical co-ordinate space with \( n \) axes \( x_1, \ldots, x_n \), and a dynamic system with variables \( x_1, \ldots, x_n \) provide a one-one correspondence between each point of the space (within some region) and each state of the system. The region is the system's 'phase-space'.

19/6. A primary operation discovers the system's behaviour by
finding how it behaves after being released from an initial state $x_1^0, \ldots, x_n^0$. It generates one line of behaviour.

The field of a system is its phase-space filled with such lines of behaviour.

19/7. If, on repeatedly applying primary operations to a system, it is found that all the lines of behaviour which follow an initial state $S$ are equal, and if a similar equality occurs after every other initial state $S', S'', \ldots$ then the system is **regular**.

Such a system can be represented by equations of form

$$
\begin{align*}
    x_1 &= F_1(x_1^0, \ldots, x_n^0; t) \\
    \ldots \ldots \ldots \\
    x_n &= F_n(x_1^0, \ldots, x_n^0; t)
\end{align*}
$$

Obviously, if the initial state is at $t = 0$, we must have

$$
F_i(x_1^0, \ldots, x_n^0; 0) = x_i^0 \quad (i = 1, \ldots, n).
$$

The equations are the written form of the lines of behaviour; and the forms $F_i$ define the field. They are obtained directly from the results of the primary operations.

19/8. If, on repeatedly applying primary operations to a system, it is found that all lines of behaviour which follow a state $S$ are equal, no matter how the system arrived at $S$, and if a similar equality occurs after every other state $S', S'', \ldots$ then the system is **absolute**.

19/9. A system is ‘state-determined’ if the occurrence of a particular state is sufficient to determine the line of behaviour which follows. Reference to the preceding section shows that absolute systems are state-determined, and vice versa.

The equations of an absolute system form a group

19/10. **Theorem.** That the equations

$$
\begin{align*}
    x_i &= F_i(x_1^0, \ldots, x_n^0; t) \quad (i = 1, \ldots, n)
\end{align*}
$$

should be those of an absolute system, it is necessary that, regarded as a substitution converting $x_1^0, \ldots, x_n^0$ to $x_1, \ldots, x_n$,
they should form a finite continuous (Lie) group of order one with \( t \) as parameter.

(1) The system is assumed absolute. Let the initial state of the variables be \( x^0 \), where the single symbol represents all \( n \), and let time \( t' \) elapse so that \( x^0 \) changes to \( x' \). With \( x' \) as initial state let time \( t' \) elapse so that \( x' \) changes to \( x'' \). As the system is absolute, the same line of behaviour will be followed if the system starts at \( x^0 \) and goes on for time \( t' + t'' \). So

\[
\begin{align*}
x'' &= F(x'_1, \ldots, x'_n; \; t') = F(x^0_1, \ldots, x^0_n; \; t' + t'') \\
\text{but} \quad x'_i &= F_i(x^0_1, \ldots, x^0_n; \; t') \quad (i = 1, \ldots, n)
\end{align*}
\]

for all values of \( x^0, t' \) and \( t'' \) over some given region. The equation is known to be one way of defining a one-parameter finite continuous group.

(2) The group property is not, however, sufficient to ensure absoluteness. Thus consider \( x = (1 + t)x^0 \); the times do not combine by addition, which has just been shown to be necessary.

**Example:** The system with lines of behaviour given by

\[
\begin{align*}
x_1 &= a_1^0 + a_2^0 t + t^2 \\
x_2 &= a_2^0 + 2t
\end{align*}
\]

is absolute, but the system with lines given by

\[
\begin{align*}
x_1 &= a_1^0 + a_2^0 t + t^2 \\
x_2 &= a_2^0 + t
\end{align*}
\]

is not.
The canonical equations of an absolute system

19/11. **Theorem**: That a system \( x_1, \ldots, x_n \) should be absolute it is necessary and sufficient that the \( x \)'s, as functions of \( t \), should satisfy differential equations

\[
\begin{align*}
\frac{dx_1}{dt} &= f_1(x_1, \ldots, x_n) \\
& \quad \ldots \\
\frac{dx_n}{dt} &= f_n(x_1, \ldots, x_n)
\end{align*}
\]  

(1)

where the \( f \)'s are single-valued, but not necessarily continuous, functions of their arguments; in other words, the fluxions of the set \( x_1, \ldots, x_n \) can be specified as functions of that set and of no other functions of the time, explicit or implicit.

(The equations will be written sometimes as shown, sometimes as

\[
\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n)
\]

(2)

and sometimes abbreviated to \( \dot{x} = f(x) \), where each letter represents the whole set, when the context indicates the meaning sufficiently.)

(1) Start the absolute system at \( x_1^0, \ldots, x_n^0 \) at time \( t = 0 \) and let it change to \( x_1, \ldots, x_n \) at time \( t \), and then on to \( x_1 + dx_1, \ldots, x_n + dx_n \) at time \( t + dt \). Also start it at \( x_1, \ldots, x_n \) at time \( t = 0 \) and let time \( dt \) elapse. By the group property (S. 19/10) the final states must be the same. Using the same notation as S. 19/10, and starting from \( x_i^0, \) \( x_i \) changes to \( F_i(x^0; t + dt) \) and starting at \( x_i \) it gets to \( F_i(x; dt) \).

Therefore

\[
F_i(x^0; t + dt) = F_i(x; dt)
\]

(3)

\( (i = 1, \ldots, n) \).

Expand by Taylor's theorem and write \( \partial \partial F_i(a; b) \) as \( F_i(a; b) \).

Then

\[
F_i(x^0; t + dt) = F_i(x; 0) + dt \cdot F_i(x; 0)
\]

(4)

\( (i = 1, \ldots, n) \)

But both \( F_i(x^0; t) \) and \( F_i(x; 0) \) equal \( x_i \).

Therefore

\[
F_i(x^0; t) = F_i(x; 0)
\]

(3)

\( (i = 1, \ldots, n) \).

But

\[
\frac{dx_i}{dt} = \frac{\partial}{\partial t} F_i(x^0; t) \]

so

\[
= F_i(x^0; t)
\]

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so, by (3), \[
\frac{dx_i}{dt} = F'_i(x; 0) \quad (i = 1, \ldots, n)
\]

which proves the theorem, since \( F'_i(x; 0) \) contains \( t \) only in \( x_1, \ldots, x_n \) and not in any other form, either explicit or implicit.

**Example 1:** The absolute system of S. 19/10, treated in this way, yields the differential equations
\[
\begin{align*}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= 2
\end{align*}
\]

The second system may not be treated in this way as it is not absolute and the group property does not hold.

**Corollary:**

\( f(x_1, \ldots, x_n) = \left[ \frac{\partial}{\partial t} F(x_1, \ldots, x_n; t) \right]_{t=0} \quad (i = 1, \ldots, n) \)

(2) Given the differential equations, they may be written
\[
\frac{dx_i}{dt} = f(x_1, \ldots, x_n), \quad (i = 1, \ldots, n)
\]

and this shows that a given set of values of \( x_1, \ldots, x_n \), i.e. a given state of the system, specifies completely what change \( dx_i \) will occur in each variable \( x_i \) during the next time-interval \( dt \). By integration this defines the line of behaviour from that state. The system is therefore absolute.

**Example 2:** By integrating
\[
\begin{align*}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= 2
\end{align*}
\]

the group equations of the example of S. 19/10 are regained.

**Example 3:** The equations of the homeostat may be obtained thus:—If \( \varphi_i \) is the angle of deviation of the \( i \)th magnet from its central position, the forces acting on \( \varphi_i \) are the momentum, proportional to \( \dot{\varphi}_i \), the friction, also proportional to \( \dot{\varphi}_i \), and the four currents in the coil, proportional to \( x_1, x_2, x_3 \) and \( x_4 \). If linearity is assumed, and if all four units are constructionally identical, we have
\[
\frac{d}{dt}(m\dot{\varphi}_i) = -k\dot{\varphi}_i + l(p - q)(a_1\varphi_1 + \ldots + a_4\varphi_4) \\
(i = 1, 2, 3, 4)
\]
where $p$ and $q$ are the potentials at the ends of the trough, $l$ depends on the valve, $k$ depends on the friction at the vane, and $m$ depends on the moment of inertia of the magnet. If

$$h = \frac{l(p - q)}{m}, \quad j = \frac{k}{m}$$

then the equations can be written

$$\begin{align*}
\frac{dx_1}{dt} &= \dot{x}_1 \\
\frac{dx_i}{dt} &= \dot{x}_i + \frac{k}{m} \left( \frac{l(p - q)}{k} (a_{4i}x_1 + \ldots + a_{4i}x_4) - j\dot{x}_1 \right) \\
\frac{dx_i}{dt} &= h(a_{4i}x_1 + \ldots + a_{4i}x_4) - j\dot{x}_1 \\
\frac{dx_i}{dt} &= \dot{x}_i \\
\frac{dx_i}{dt} &= \frac{k}{m} \left( \frac{l(p - q)}{k} (a_{4i}x_1 + \ldots + a_{4i}x_4) - j\dot{x}_1 \right)
\end{align*}$$

(i = 1, 2, 3, 4)

which shows the 8-variable system to be absolute.

They may also be written

$$\begin{align*}
\frac{dx_1}{dt} &= \dot{x}_1 \\
\frac{dx_i}{dt} &= \dot{x}_i + \frac{k}{m} \left( \frac{l(p - q)}{k} (a_{4i}x_1 + \ldots + a_{4i}x_4) - \dot{x}_1 \right) \\
\frac{dx_i}{dt} &= \frac{k}{m} \left( \frac{l(p - q)}{k} (a_{4i}x_1 + \ldots + a_{4i}x_4) - \dot{x}_1 \right)
\end{align*}$$

Let $m \to 0$. $dx_i/dt$ becomes very large, but not $dx_i/dt$.

So $\dot{x}_1$ tends rapidly towards

$$\frac{l(p - q)}{k} (a_{4i}x_1 + \ldots + a_{4i}x_4)$$

while the $x_i$'s, changing slowly, cannot alter rapidly the value towards which $\dot{x}_1$ is tending. In the limit,

$$\begin{align*}
\frac{dx_1}{dt} &= \dot{x}_1 = \frac{l(p - q)}{k} (a_{4i}x_1 + \ldots + a_{4i}x_4) \\
\frac{dx_i}{dt} &= a_{4i}x_1 + \ldots + a_{4i}x_4 \\
\frac{dx_i}{dt} &= a_{4i}x_1 + \ldots + a_{4i}x_4
\end{align*}$$

(i = 1, 2, 3, 4)

Change the time-scale by $\tau = \frac{l(p - q)}{k} t$;

$$\begin{align*}
\frac{dx_1}{d\tau} &= a_{4i}x_1 + \ldots + a_{4i}x_4 \\
\frac{dx_i}{d\tau} &= a_{4i}x_1 + \ldots + a_{4i}x_4
\end{align*}$$

(i = 1, 2, 3, 4)

showing the system $x_1, \ldots, x_4$ to be absolute and linear. The $a_i$'s are now the values set by the hand-controls of Figure 8/8/3.

19/12. That a system should be absolute, it is necessary and sufficient that at no point of the field should a line of behaviour
bifurate. The statement can be verified from the definition or from the theorem of S. 19/11. The statement does not prevent lines of behaviour from running together.

19/13. The theorems of the previous four sections show that the following properties, collected for convenience, in a system $x_1, \ldots, x_n$, are all equivalent in that the possession of any one of them implies the others:

1. From any point in the field departs only one line of behaviour (S. 19/8);
2. the system is state-determined (S. 19/9);
3. the system has lines of behaviour whose equations specify a finite continuous group of order one;
4. the system has lines of behaviour specified by differential equations of form

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n) \quad (i = 1, \ldots, n)$$

where the right-hand side contains no functions of $t$ except those whose fluxions are given on the left.

19/14. From the experimental point of view the simplest test for absoluteness is to see whether the lines of behaviour are state-determined. An example has been given in S. 2/15. It will be noticed that experimentally one cannot prove a system to be absolute—one can only say that the evidence does not disprove the possibility. On the other hand, one value may be sufficient to prove that the system is not absolute.

19/15. A simple example of a system which is regular but not absolute is given by the following apparatus. A table top is altered so that instead of being flat, it undulates irregularly but gently like a putting-green (Figure 19/15/1). Looking down on it from above, we can mark across it a rectangular grid of lines to act as co-ordinates. If we place a ball at any point and then release it, the ball will roll, and by marking its position at, say, every one-tenth second we can determine the lines of behaviour of the two-variable system provided by the two co-ordinates.

If the table is well made, the lines of behaviour will be accurately reproducible and the system will be regular. Yet the experimenter, if he knew nothing of forces, gravity, or momenta,
would find the system unsatisfactory. He would establish that
the ball, started at $A$, always went to $A'$; and started at $B$ it
always went to $B'$. He would find its behaviour at $C$ difficult
to explain. And if he tried to clarify the situation by starting
the ball at $C$ itself, he would find it went to $D$! He would say
that he could make nothing of the system; for although each

![Figure 19/15/1.](image)

line of behaviour is accurately reproducible, the different lines
of behaviour have no relation to one another.

This lack of relation means that they do not form a 'group'.
But whether the experimenter agrees with this or not, he will,
in practice, reject this 2-variable system and will not rest till
he has discovered, either for himself or by following Newton,
a system that is state-determined. In my theory I insist on the
systems being absolute because I agree with the experimenter
who, in his practical work, is similarly insistent.

19/16. That the field of a system should not vary with time,
it is necessary and sufficient that the system be regular. The
proof is obvious.

19/17. One reason why a system's absoluteness is important is
because the system is thereby shown to be adequately isolated
from other unknown and irregularly varying parameters. This
demonstration is obviously fundamental in the experimental
study of a dynamic system, for the proof of isolation comes,
not from an examination of the material substance of the system (S. 14/1), which may be misleading and in any case presupposes that we know beforehand what makes for isolation and what does not, but from a direct test on the behaviour itself.

Closely related to this in a fundamental way is the fact that Shannon's concept of a 'noiseless transducer' is identical in definition with my definition of an absolute system. Thus he defines such a transducer as one that, having states \( x \) and an input \( x \), will, if in state \( z_n \) and given input \( x_n \), change to a new state \( z_{n+1} \) that is a function only of \( x_n \) and \( z_n \):

\[
z_{n+1} = g(x_n, z_n)
\]

Though expressed in a superficially different form, this equation is identical with my 'canonical' equation, for it says simply that if the parameters \( x \) and the state of the system are given, then the system's next step is determined. Thus the communication engineer, if he were to observe the physicist and the psychologist for the first time, would say that they seem to prefer to work with noiseless systems. His remark would not be as trite as it seems, for from it flow far-reaching consequences and the possibilities of rigorous deduction.

19/18. A second feature which makes absoluteness important is that its presence establishes, by appeal only to the behaviour, that the system of variables is complete, i.e. that it includes all the variables necessary for the specification of the system.

19/19. When we assemble a machine, we usually know the canonical equations directly. If, for instance, certain masses, springs, magnets, be put together in a certain way the mathematical physicist knows how to write down the differential equations specifying the subsequent behaviour.

His equations are not always in our canonical form, but they can always be converted to this form provided that the system is isolated, i.e. not subjected to arbitrary interference, and is determinate.

19/20. In general there are two methods for studying a dynamic system. One method is to know the properties of the parts and the pattern of assembly. With this knowledge the canonical
equations can be written down, and their integration predicts the
behaviour of the whole system. The other method is to study
the behaviour of the whole system empirically. From this
knowledge the group equations are obtained: differentiation of
the functions then gives the canonical equations and thus the
relations between the parts.

Sometimes systems that are known to be isolated and complete
are treated by some method not identical with that used here.
In those cases some manipulation may be necessary to convert
the other form into ours. Some of the possible manipulations
will be shown in the next few sections.

19/21. Systems can sometimes be described better after a change
of co-ordinates. This means changing from the original variables
\( x_1, \ldots, x_n \) to a new set \( y_1, \ldots, y_n \), equal in number to the
old and related in some way

\[
y_i = \phi(x_1, \ldots, x_n) \quad (i = 1, \ldots, n)
\]

If we think of the variables as being represented by dials, the
change means changing to a new set of dials each of which
indicates some function of the old. It is easily shown that such
a change of co-ordinates does not change the absoluteness.

19/22. In the 'homeostat' example of S. 19/11 a derivative
was treated as an independent variable. I have found this
treatment to be generally advantageous: it leads to no difficulty
or inconsistency, and gives a beautiful uniformity of method.

For example, if we have the equations of an absolute system
we can write them as

\[
\dot{x}_i - f(x_1, \ldots, x_n) = 0 \quad (i = 1, \ldots, n)
\]
treating them as \( n \) equations in \( 2n \) algebraically independent
variables \( x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n \). Now differentiate all
the equations \( q \) times, getting \((q + 1)n\) equations with \((q + 2)n\)
variables and derivatives. We can then select \( n \) of these vari-
ables arbitrarily, and noticing that we also want the next higher
derivatives of these \( n \), we can eliminate the other \( qn \) variables,
using up \( qn \) equations. If the variables selected were \( z_1, \ldots, z_n \),
we now have \( n \) equations, in \( 2n \) variables, of type

\[
\phi(z_1, \ldots, z_n, \dot{z}_1, \ldots, \dot{z}_n) = 0 \quad (i = 1, \ldots, n)
\]
These have only to be solved for $\dot{z}_1, \ldots, \dot{z}_n$ in terms of $z_1, \ldots, z_n$ and the equations are in canonical form. So the new system is also absolute.

This transformation implies that in an absolute system we can avoid direct reference to some of the variables provided we use derivatives of the remaining variables to replace them.

**Example:**

\[
\begin{align*}
\dot{x}_1 &= x_1 - x_2 \\
\dot{x}_2 &= 3x_1 + x_2
\end{align*}
\]

can be changed to omit direct reference to $x_2$ by using $\dot{x}_1$ as a new independent variable. It is easily converted to

\[
\begin{align*}
\frac{dx_1}{dt} &= \dot{x}_1 \\
\frac{d\dot{x}_1}{dt} &= -4x_1 + 2\dot{x}_1
\end{align*}
\]

which is in canonical form in the variables $x_1$ and $\dot{x}_1$.

**19/23.** Systems which are isolated but in which effects are transmitted from one variable to another with some finite delay may be rendered absolute by adding derivatives as variables. Thus, if the effect of $x_1$ takes 2 units of time to reach $x_2$, while $x_2$'s effect takes 1 unit of time to reach $x_1$, and if we write $\alpha(t)$ to show the functional dependence,

\[
\frac{dx_1(t)}{dt} = f_1(x_1(t), x_2(t - 2))
\]

\[
\frac{dx_2(t)}{dt} = f_2(x_1(t - 1), x_2(t)).
\]

This is not in canonical form; but by expanding $x_1(t - 1)$ and $x_2(t - 2)$ in Taylor's series and then adding to the system as many derivatives as are necessary to give the accuracy required, we can obtain an absolute system which resembles it as closely as we please.

**19/24.** If a variable depends on some accumulative effect so that, say, $\dot{x}_1 = \int_{a}^{t} f(x_3)dt$, then if we put $\int_{a}^{t} f(x_3)dt = y$, we get
the equivalent form
\[
\frac{dx_1}{dt} = f(y)
\]
\[
\frac{dy}{dt} = \phi(x_2)
\]
\[
\frac{dx_2}{dt} = \ldots \text{ etc.}
\]
which is in canonical form.

19/25. If a variable depends on velocity effects so that, for instance
\[
\frac{dx_1}{dt} = f_1\left(\frac{dx_2}{dt}, x_1, x_2\right)
\]
\[
\frac{dx_2}{dt} = f_2(x_1, x_2)
\]
then if we substitute for \(\frac{dx_2}{dt}\) in \(f_1(\ldots)\) we get the canonical form
\[
\frac{dx_1}{dt} = f_1\left(f_2(x_1, x_2), x_1, x_2\right)
\]
\[
\frac{dx_2}{dt} = f_2(x_1, x_2)
\]

19/26. If one variable changes either instantaneously or fast enough to be so considered without serious error, then its value can be given as a function of those of the other variables; and it can therefore be eliminated from the system.

19/27. Explicit solutions of the canonical equations
\[
\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n) \quad (i = 1, \ldots, n)
\]
will seldom be needed in our discussion, but some methods will be given as they will be required for the examples.

(1) A simple symbolic solution, giving the first few terms of
\(x_i\) as a power series in \(t\), is given by
\[
x_i = e^{X}x_i^0 \quad (i = 1, \ldots, n) \quad . \quad . \quad (1)
\]
where \(X\) is the operator
\[
f_i(x_1^0, \ldots, x_n^0) \frac{\partial}{\partial x_1} + \ldots + f_n(x_1^0, \ldots, x_n^0) \frac{\partial}{\partial x_n} \quad . \quad . \quad (2)
\]
and
\[
e^{X} = 1 + tX + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 + \ldots \quad . \quad (3)
\]
It has the important property that any function \( \Phi(x_1, \ldots, x_n) \) can be shown as a function of \( t \), if the \( x \)'s start from \( x_1^0, \ldots, x_n^0 \), by
\[
\Phi(x_1, \ldots, x_n) = e^{\lambda t} \Phi(x_1^0, \ldots, x_n^0).
\]

(2) If the functions \( f_i \) are linear so that
\[
\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_1
\]
\[
\ldots \ldots \ldots
\]
\[
\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_n
\]
then if the \( b \)'s are zero (as can be arranged by a change of origin) the equations may be written in matrix form as
\[
\dot{x} = Ax
\]
where \( \dot{x} \) and \( x \) are column vectors and \( A \) is the square matrix \([a_{ij}]\). In matrix notation the solution may be written
\[
x = e^{At}x^0
\]

(3) Most convenient form of the linear form is the recently developed method of the Laplace transform. The standard text-books should be consulted for details.

19/28. Any comparison of an absolute system with the other types of system treated in mechanics and in thermodynamics must be made with caution. Thus, it should be noticed that the concept of the absolute system makes no reference to energy or its conservation, treating it as irrelevant. It will also be noticed that the absolute system, whatever the ‘machine’ providing it, is essentially irreversible. This can be established either by examining the group equations of S. 19/10, the canonical equations of S. 19/11, or, in a particular case, by examining the field of the common pendulum in Figure 2/15/1.

Reference