Reification as the Birth of Metaphor*

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"Basically, I'm not interested in doing research and I never have been. I'm interested in *understanding*, which is quite a different thing." — David Blackwell, referring to his work as a mathematician [Albers & Alexanderson, 1985, p. 19]

1. Introduction: the elusive experience of understanding

The experience of understanding is doubly elusive: it is difficult to achieve and to sustain, and it is even more difficult to capture and to explain. I can clearly remember the event which, for the first time, made me aware of the degree of my ignorance in this respect. I was a beginning teacher and I discovered to my surprise that students who had a good command over systems of linear equations might still be unable to deal with such questions as, "For what value of a parameter q does the given system of linear equations have no solution?". I approached the difficulty casually, confident that we would overcome the problem in an hour or two. My expectations proved wrong. It took days until I felt that the class could cope with parameters. But even then the situation was not what I had hoped for: at the final test only one student managed to produce fully satisfactory solutions to all the problems I posed. In a private conversation with him I remarked: "It seems that you are the only one in this class who really understood the subject." To my distress, the praise was greeted with an angry response: "Wrong! I didn't understand anything. I did what I did but I don't know why it worked". I tried to prove him wrong. I presented him with several other problems, one quite unlike the other, and he solved all of them without a visible difficulty. I claimed that this kind of question just cannot be answered by mechanical application of an algorithm. He kept insisting that he "did not understand anything". We ended up frustrated and puzzled. He felt he did not understand parameters; I sensed that I did not understand understanding.

Reflections on my own experience helped, but only to some extent. I could remember myself as a graduate mathematics student passing exams without difficulty but often feeling that the ease with which I was doing things was not enough to give me the sense of true understanding. Some time later I was happy to find out that even people who grew up to become well-known mathematicians were not altogether unfamiliar with this kind of experience. For example, Paul Halmos [1985] recalls in his "automatography":

> ... I was a student, sometimes pretty good and sometimes less good. Symbols didn't bother ne. I could juggle them quite well ...[but] I was stumped by the infinitesimal subtlety of epsilonic analysis. I could read analytic proofs, remember them if I made an effort, and reproduce them, sort of, but I didn't really know what was going on. [p. 47]

Halmos was fortunate enough to eventually find out what the "real knowing" was all about [Albers & Alexanderson, 1985, p. 123]:

> ... one afternoon something happened. I remember standing at the blackboard in Room 213 of the mathematics building talking with Warren Ambrose and suddenly I understood epsilon. I understood what limits were, and all of that stuff that people were drilling in me became clear. I sat down that afternoon with the calculus textbook by Granville, Smith, and Longley. All of that stuff that previously had not made any sense became obvious ...

Clearly, what people call "true" understanding must involve something that goes beyond the operative ability of solving problems and of proving theorems. But although a person may have no difficulty in diagnosing the degree of his or her understanding, he or she does not find it equally easy to name the criteria according to which such assessment is made. Many works have already been written in which an attempt is made to understand what understanding is all about (for a comprehensive and insightful survey of these see Sierpinska [1993]). In the present paper I will try to take another little step toward capturing the gist of this elusive something that makes us feel we have grasped the essence of a concept, a relation, or a proof.

Let me begin with a few words on the way in which I tackled the question. My quest for a better understanding of mathematical understanding went in two directions. First, I tried to find out what contemporary thinkers have to say on the subject. I soon discovered that, as far as the issue of understanding is concerned, current developments in the psychology of mathematics go hand-in-hand with some of the most significant recent advances in linguistics and in philosophy. The applicability of the latter to the field of mathematical education had already been noted by some researchers [e.g. Dörfler, 1991; Presmeg, 1992]. In this paper I will show how the idea of reification — the basic notion of the conceptual framework on which I have been working for quite a long time now — combines with the new general theories of understanding. I hope to make it clear that the theory of reification is perfectly in tune with the latest philosophical and linguistic developments, and that much may be gained by tightening the links between the different fields. Such a marriage of ideas will be the central theme of the next section.

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My second move was to approach the people who — so I believed — could provide me with first-hand information about the experience of understanding mathematics. I turned to research mathematicians. While choosing mathematicians as my subjects I was fully aware that this decision had some pitfalls. To begin with, I knew that what I would find in my subjects would not necessarily be generally true. Whatever the actual difference between "professionals" and "laymen", however, it was my deep conviction that mathematicians' reflections on their own thinking might provide me with insights the importance of which would go beyond the question of "professional" understanding. Another difficulty had to do with the method I chose for collecting the data. Introspection, being inherently subjective, is not necessarily the best way to obtain reliable information. However, since I was interested in the inner sensations related to the process of understanding rather than in any visible behaviors, I could think of no better means of inquiry into my interlocutors' meta-cognitive skills.

2. What the non-objectivist theory of meaning has to say

Be the problem of mathematical comprehension as unique as it may be, the issue of understanding is certainly not limited to mathematics and it would thus be a mistake to deal with the particularities of the special case in question without first referring to the existing, quite rapidly developing, general theories of meaning. The fact that I said "meaning" rather than "understanding" was not a slip of tongue. As I will try to explain, my reasons for stressing the aspect of meaning goes well beyond the obvious fact that meaning and understanding are intimately related. It is the relatively new approach to human thinking, imagination, and comprehension promoted by such writers as Lakoff [1987] and Johnson [1987] which compels me to treat the question of understanding as almost equivalent to the question of meaning.

This last statement, ostensibly not very far from the centuries-old consensus, would, in fact, stir protests among those philosophers and linguists whom Johnson classifies as "Objectivist". Objectivism, in this case, is a generic name for all those schools and theories which, either implicitly or explicitly, endorse the assumption that meaning is primarily a characteristic of signs and concepts, that it is a kind of externally determined "cargo" carried by symbols and sentences. This simple basic belief proved to be powerful enough to give rise to an all-embracing paradigm — a paradigm so broad that almost all the past and many recent schools of thought fall within its boundaries.

The basic tents of the Objectivist outlook may be summarized in a few sentences. According to Johnson's critical account, Objectivism is grounded in the view that meanings are "disembodied": they are received by a human mind rather than shaped by it. Accordingly, understanding is conceived as "grasping the meaning" and thus as a process which mediates between an individual mind and the universally experienced, absolute, ahistorical realm of facts and ideas. To put it in a different language, understanding consists in building links between symbols and a certain mind-independent reality. Further, Objectivism presupposes that all knowledge is of a propositional nature; that is, it is "conceptually and propositionally expressible in literal terms that can correspond to objective aspects of reality" [Johnson, 1987, p. 5]. Finally, the Objectivist paradigm is intimately related to the *representational view of mind* [Putnam, 1988] according to which "To know is to represent accurately (in one's head) what is outside the mind" [Rorty, 1979, p. 3).

The basic beliefs of Objectivism, even if not stated explicitly, underlie nearly all the classical theories of meaning and understanding, whatever their philosophical underpinnings. Even Internationality, with its roots in the works of Kant, Husserl, and Frege and with its current elaboration in the writings by Searle [1983], is seen by Johnson as not entirely free from "Objectivist leanings". Incidentally, Objectivism should not be mistaken for Realism and, by the same token, anti-Objectivism should not be interpreted as a denial of the existence of an objective reality. Objectivism, as conceived by Johnson, addresses the question of the way human beings create their understanding of the world rather than the question of the existence or the nature of this world.

The objectivist paradigm has been under growing criticism for the last several decades and is now being gradually abandoned by philosophers, linguists, and cognitive psychologists. Philosophers of science (Carnap, Kuhn, Feyrabend) may be those who gave the first decisive blow to the idea of a "God's eye view" of reality. In cognitive psychology, a pressing reason for a thorough revision of our views on meaning and understanding is an obvious discrepancy between Objectivism and the broadly adopted constructivist approach to learning. The anti-Objectivist trend is strongly felt also in mathematics education:

> Given that mathematics educators almost universally accept that learning is a constructive process, if is doubtful if any take a representational view literally and believe that learning is a process of immaculate perception [Cobb *et al.*, 1992, p. 3].

One way to deal with this apparent dissonance is to reverse the Objectivist version of the relationship between meaning and understanding: while Objectivism views understanding as somehow secondary to, and dependent on, predetermined meanings, non-Objectivism implies that it is our understanding which fills signs and notions with their particular meaning. While Objectivists regard meaning as a matter of a relationship between symbols and a real world and thus as quite independent of the human mind, the non-objectivist approach suggests that there is no meaning beyond that particular sense which is conferred on the symbols through our understanding.

In this view the question of the primary sources of our understanding arises. Moreover, if the meaning lies in the eyes of the beholder, should it be regarded as an entirely subjective matter? The way in which Lakoff and Johnson answer both these questions is where their truly original, imaginative contribution to the theory of meaning and understanding may be found. They devote much of their writings to a thorough description of a well-defined mechanism which turns even the most abstract of ideas into concepts filled with meaning. The possibility of shared, as-if-objective meanings stems from the fact that the functioning of this mechanism is subject to certain universal laws.

Whereas the "disembodiment" of meaning is the central motive of the Objectivist approach, Lakoff and Johnson takes it upon themselves "to put the body back into the mind". The pivotal idea of their theory is that our bodily experience is the main — in fact the only — source of understanding. In this context, explains Johnson, "experience" is to be understood in a very rich, broad sense as including basic perceptual, motor-program, emotional, historical, social, and linguistic experience" [Johnson, 1987, p. xvi]. The physical and the experimental is the basis for even the purest, most sophisticated, figments of our imagination. Moreover, even our reasoning is or can be related to primary perceptions. A careful look at the basic rules of inference will reveal that they have their roots in the physical experience of containment (being "in" or "out" of a certain space or set).

All this being said, the basic question is how "the "bodily" works its way up to the "conceptual" and "rational"." [ibid, p. xxi]. In the centre of the answer given by Lakoff and Johnson stands a metaphor. Our conceptual system is a product of metaphors which transfer the bodily experience into the less concrete realm of ideas. According to the classical definition, metaphor is a mapping from one conceptual domain to another. In this sense, the claim about a metaphorical projection from the perceptual to the abstract would seem rather implausible, as it would suggest that there is a straightforward correspondence, based on a similarity relation, between physical experiences and abstract concepts. In Lakoff and Johnson's theory, however, the term "metaphor" is used in a much broader sense than in traditional linguistics: it is not just a rhetorical form or a semantic gimmick making use of ready-make concepts. It is a mental construction which plays a constitutive role in structuring our experience and in shaping our imagination and reasoning. In other words, rather than being a product of a comparison between two existing things or ideas, metaphor, as conceived by Lakoff and Johnson, is what *brings abstract concepts into being.*

Endowed with this new interpretation of the idea of "figurative projection", let us turn to the claim about perceptual, bodily roots of all our thinking and to the role of metaphors in "putting the body into the mind". In a closer look, we will soon discover that bodily metaphors leave their traces everywhere — and above all, in our everyday language. To begin with, we talk about understanding in such perceptual terms as "I see" or "I grasp". This implies that the act of understanding is conceived as if it was an act of absorbing physical stimuli. Further, let us consider such expressions as "burning love" or "cognitive strain". Saying that they are founded in such comparisons as "love is like a fire" or "mind is like a container" would not do justice to the power of metaphor. In fact, there may be little to compare *before* the links exposed by these expressions are established. These links have a power of creating meanings. It is thanks to them and to the fact that they refer to abstract ideas (love, mind) and to things with which we are intimately familiar through our perceptual experience (fire, containment) that such terms as "love" and "mind" become clearly delineated and thus meaningful to us. For instance, the message of coupling "love" with "burning" is that love may "warm our hearts" (another metaphor!), but it may also have a devastating effect, like fire often has. In fact, we don't even have to elicit metaphorical connections through language in order to conceive of abstract concepts in terms of perceptual experiences. To put it in a somewhat simplistic but persuasive (and metaphorical!) way, the direct immediate understanding — the basic understanding achieved through perception — produces the primitives from which more advanced meanings are built.

Another question that requires thorough analysis concerns the mechanism of metaphorical construction. According to Lakoff and Johnson, the vehicle which carries our experimentally constructed knowledge is an *embodied schema* (known also as an *image schema*). Johnson defines embodied schemata as "structures *of an activity* by which we organize our experience in ways that we can comprehend. They are a primary means by which we *construct* or *constitute* order and are not mere passive receptacles into which experience is poured." [pp. 29-30] These structures underlie our ability to abstract and generalize: they are, in a sense, the bare bones of our experience, the flesh of the concrete instantiations having been stripped away. Throughout our conscious existence we are engaged in the continuous activity of putting order onto our manifold interactions with the world, in the never-ending attempt to make a sense of the things we experience. "A schema is a recurrent pattern, shape, and regularity in, or of, these ongoing (ordering) activities". [p. 29] To sum up, an embodied schema is what epitomizes, organizes, and preserves "for future use" the essence of our experience and, as such, it is our tool for handling the multifarious physical and intellectual stimuli with which we are faced throughout our lives.

Unlike the symbolic and linguistic expressions with which we communicate our knowledge to others, and quite contrary to the Objectivist vision of knowledge, embodied schemata are usually *non-propositional.* This central characteristic is already reflected in the name given to these special mental constructs: they are image-like and embodied, embodied in the sense that they should be viewed as analog reflections of bodily experience rather than as factual statements we may wish to check for validity. The non-propositional nature of embodied schemata makes it difficult, sometimes impossible, to describe them in words. Only their entailments — the pieces of factual knowledge generated by the schemata — are amenable to verbal presentation. As to the schemata themselves, well, "while we may describe features of their structure propositionally using finite representations, we thereby lose our ability to explain their natural operation and transformations." [ibid, p. 23] We should keep this issue in mind while discussing the difficulty invariably experienced by mathematicians who try to communicate their highly abstract ideas to others.

If embodied schemata cannot be viewed as the mental counterpart of a system of factual statements, the question arises about the cognitive means by which such schemata are handled. Here again, misled by our previous knowledge, we may easily slip into an oversimplified, distorted version. Mental images seem to be the natural alternative to the propositional structure. The idea that an embodied schema is, in fact, a mental image is even more convincing in view of the fact that both these cognitive structures have the same leading characteristics: they are *analog* and *holistic*. True, an embodied schema may be buttressed by a mental image, but there is a crucial difference between the two: whereas a mental image is always an image of something concrete and is therefore full of details (that is why Johnson calls it a "rich image"), an embodied schema is general and malleable. It is but a skeleton with many variable parts which, being undetermined, cannot be visualized. The generality of this embodied schema is what gives it its structuring power and its capacity to encompass in one manageable mental construction a wide variety of our experiences. (In spite of an almost unbridgeable gap between Lakoff and Johnson's theory and the information-processing approach to cognition, one may be tempted to compare the idea of embodied schema to Minsky's [1975] concept of frame.)

The initial question about how bodily experience is metaphorically transmitted into a sphere of more abstract thinking has now got its answer: embodied schemata, originally built to put order onto our physical experience, are "borrowed" to give shape, structure, and meaning to our imagination.

The constitutive role of metaphor in scientific thinking has been acknowledged for some time now [see e.g. Ortony, 1979; Knorr, 1980]. Recently, some writers have introduced the concept of metaphor (and metonymy) to their analysis of mathematical thinking [see e.g. Pimm, 1987, 1990]. Lakoff and Johnson's theory, however, differs from all the previous works in two respects. First, it exceeds all the other approaches in the importance it ascribes to metaphors and to their impact on human thinking. Lakoff and Johnson's central thesis is that metaphors constitute the universe of abstract ideas, that they create rather than reflect it, that they are the source of our understanding, imagination, and reasoning. Second, the focus of the theory is on a special kind of metaphor — a metaphor the source of which is in our bodily experience. Thus, Lakoff and Johnson's central claim is that abstract ideas inherit the structure of physical, bodily, perceptual, experience. In the next sections I will try to translate these ideas into the domain of mathematics. This special context will demonstrate with particular clarity that as far as our imagination is concerned, the mechanism of metaphorical construction is a double-edged sword. On the one hand, it is what brings the universe of abstract ideas into existence in the first place; on the other hand, however, "the metaphors we live by" put obvious constraints on our imagination and understanding. Our comprehension and fantasy can only reach as far as the existing metaphorical structures allow. Creative mathematicians, in order to make any progress, must often break beyond the demarcation line drawn by bodily experience.

3. The origins of mathematical objects

The idea that mathematical abstractions are tightly connected to, and constrained by, the knowledge we construct through our encounters with physical reality has already been raised by several authors. "[W]e can identify a perceptual basis for mathematical knowledge" asserts Kitcher [1984, p. 11], while launching a defense of the (quasi-)empiricist vision of mathematics. Cobb [1985, 1990] substantiates the claim that "actual and represented sensory-motor action plays a crucial role in mathematical activity" by showing how concrete actions lead to the emergence of the concept of number in young children.

In advanced mathematics, at levels far removed from physical reality, it may well be that the immediate source of a basic metaphor is another, lower-level mathematical structure. Even so, and however long the chain of metaphors may be, whatever is going on in our mind is primarily rooted in our body. The intelligibility of abstract objects stems from their being metaphorical reflections of our bodily experience. It will be my goal in the following discussion to explicate the nature of the relationship between the abstract and the experiential and to show how the bodily aspects of our existence both enable and constrain our understanding. Mathematicians' accounts of their own quest after meaning will be the principal source of evidence. I'll confine myself to the material collected during three full-length (three hours and more) semi-structured interviews with renowned mathematicians: a logician (let us call him ML), a set-theorist (ST), and a specialist on ergodic theory (ET) (all the interviews were conducted in Hebrew). I will also resort, here and there, to mathematicians' autobiographical writings.

3.1 Mathematical concepts with a human face: the metaphor of ontological object

The mathematical universe, populated by mathematical objects and animated by the manipulations which may be performed on these objects, can hardly be understood in any other way than as a metaphorical reflection of a physical world. Lakoff and Johnson [1980] explain the special strength of the "metaphor of an ontological object":

> Our experience of physical objects and substances provides a further basis for understanding ... Understanding our experiences in terms of objects and substances allows us to pick out parts of our experience and treat them as discrete entities or substances of a uniform kind. Once we can identify our experiences as entities or substances, we can refer to them, categorize them, group them, and quantify them — and, by this means, reason about them. [p. 25]

Listening to mathematicians talking about their ideas is enough to make one realize that in mathematics, the metaphor of ontological object is ubiquitous. To begin with, the language used in textbooks to describe the basic mathematical entities is clearly object-oriented: "a complex number is an ordered pair of ...", "a group is a set of elements together with a binary operation such that ...", "let's take a bounded region of an n-dimensional space ...". The names given to different mathematical entities and properties clearly have their roots in the world of material objects: a function may be *increasing* or *decreasing,* a field may be *closed* or *open*, a model or a theory may be *saturated* or *stable*. The fact that we use the word "existence" with reference to abstract objects (as in existence theorems) reflects in the most persuasive way the metaphorical nature of the world of abstract ideas. Greeno [1991] makes the metaphor of ontological object explicit when he compares understanding mathematics to "knowing one's way around in an environment and knowing how to use its resources." [p. 175]

Metaphorical motifs appeared time and again in my conversations with mathematicians. In the answer to the question about what happens in their minds when they feel that they have arrived at a deep understanding of a mathematical idea, they unanimously claimed that the basis of this unique feeling is not a manipulative power but an ability to "identify a structure that [one is] able to grasp somehow" (ST), or "to see an image" (ET), or "to play with some unclear images of things" (ML). To put it in ET's words: "In those regions where I feel an expert, ...the concepts, the [mathematical] objects turned tangible for me." ST expressed his need for a metaphor explicitly (ST used the word "metaphor" on his own accord; needless to say, I tried to formulate questions to the interviewees in a theory-free language; at that particular stage it was not too difficult, as the idea of applying Lakoff and Johnson's framework to the analysis of mathematicians' understanding imposed itself on me as a result of the interviews):

> To understand a new concept I must create an appropriate metaphor. A personification. Or a spatial metaphor. A metaphor of structure. Only then can I answer questions, solve problems. I may even be able then to perform some manipulations on the concept. Only when I have the metaphor. Without the metaphor I just can't do it.

He proceeded with a description which left no doubt as to the bodily origins of the metaphors he had in mind. First, there was a spatial metaphor:

> In the structure, there are spatial elements. Many of them. It's strange, but the truth is that my student also has noticed it ... a great many spatial elements. And we are dealing here with the most abstract things one can think about! Things that have nothing to do with geometry, [that are] devoid of anything physical ... The way we think is always by means of something spatial ... Like in "This concept is above this one" or "Let's move along this axis or along the other one". There are no axes in the problem and still ...

I quote ST here, but, in fact, all three interviewees stressed the importance of "seeing a structure". It would be appropriate here to emphasize the delicate distinction between the issue of seeing a structure and having a metaphor. While structure is an inherent characteristic of every subject matter and, as such, may be considered independently, without reference to any external elements, a metaphor is what ties a given idea to concepts with which a person is already familiar. Thus, an ability to see a structure is not enough to gain an understanding. ST told me a story which aptly illustrates this claim.

> Halmos listened once to my lecture. He is not in set theory (now), he doesn't keep abreast of the progress in the domain and he doesn't know the concepts. After my lecture he told me, "I didn't understand anything, but I enjoyed your talk greatly." "How come?" I asked. And he said: "I don't know the concepts, I don't know what they say, but the structure — how they relate one to another — I grasped very well indeed."

Spatial thinking is not the only way to conceive of structure. ST told me about yet another kind of metaphor which appears in his mathematical reasoning: a personification. "Perhaps the most obvious ontological metaphors are those where the physical object is further specified as being a person", observed Lakoff and Johnson [1980, p. 33]. Hadamard [1949] was probably the first to notice that a mathematical concept may sometimes be imagined as having a "human face" — "a physiognomy which allows us to think of it as a unique thing, however complicated it may be, just as we see a face of a man". ST gave an even more colorful description:

> There is, first and foremost, an element of personification in mathematical concepts ... for example yesterday, I thought about some coordinates ... (I told myself) "this coordinate moves here and ... it commands this one to do this and that." There are elements of animation. It's not geometric in the sense of geometric pictures, but you see some people moving and talking to each other.

In a similar vein, ML remarked:

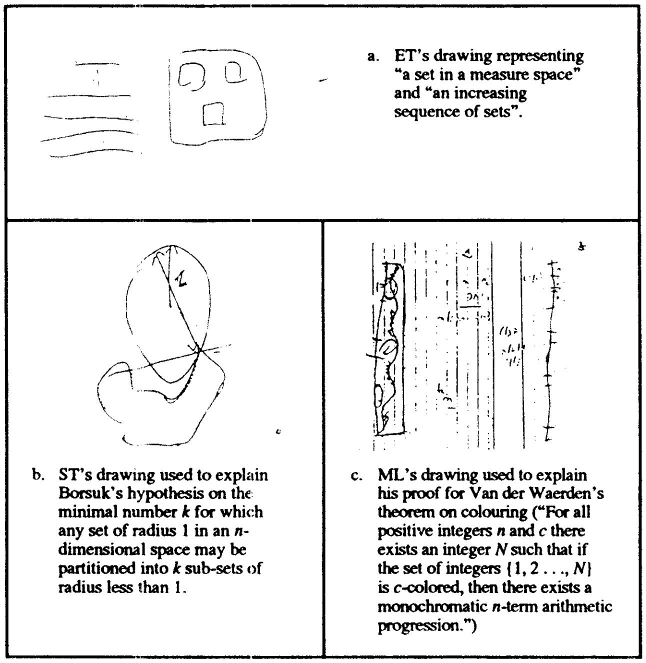
> When I think about a fat man, I see (in my mind's eye) a fat man. Saturated model seems to me quite like that — like a padded guy.

The way mathematicians refer to the mental constructs with which they pave their way toward understanding often brings to mind the concept of an embodied schema — the carrier of a metaphor. For example, Hadamard's term "cloudy imagery" is more aptly interpreted as evidence for the appearance of embodied schemata than as a reference to a simple visualization. Hadamard himself uses the word "schema" to describe this particular mental construct:

> "...every mathematical research compels me to build ... a schema, which is always and must be *of a vague character* [my emphasis, A.S.], so as not to be deceptive." [p. 77]

The "vague character" is the leading characteristic thanks to which the embodied schema acquires its generality and its unifying power. According to Johnson, this is exactly the feature which is lacking in a "rich image" — namely, a simple visualization.

Even though all my interviewees remarked many times that they frequently resort to visualization [compare Dreyfus, 1991], they also stressed that pictures, whether mental or in the form of drawings, are only a part of the story. They support thinking, but they do not reflect it in all dimensions. Using the term introduced by Dörfler, I would say that images of any kind are but *concrete carriers* for the embodied schemata. The pictures mathematicians use to draw on a paper or on a blackboard serve a double purpose: they are "something to think with" and they function as a means of communication. In spite of the obvious limitations of a picture as an expression of generality, both aspects are extremely important.



a. ET's drawing representing "a set in a measure space" and "an increasing sequence of sets".

b. ST's drawing used to explain Borsuk's hypothesis on the minimal number k for which any set of radius 1 in an n-dimensional space may be partitioned into k sub-sets of radius less than 1.

c. ML's drawing used to explain his proof for Van der Waerden's theorem on colouring ("For all positive integers n and c there exists an integer N such that if the set of integers $\{1, 2 \ldots, N\}$ is c-colored, then there exists a monochromatic n-term arithmetic progression.")

Figure 1
Drawings produced by interviewees to illustrate mathematical ideas

All the mathematicians I talked to said they just could not think without making pictures. All of them drew different shapes when trying to explain to me certain mathematical theorems or conjectures. Examples of the pictures they produced are presented in Figure 1.

3.2 What it means to say a person is "intimately" familiar with a mathematical concept and how such familiarity affects reasoning

The most natural way to assess one's understanding of a mathematical idea is to estimate the ease with which one reasons and discovers new facts about it. On the face of it, mathematical reasoning is always based on a sequence of inferences which, in a systematic way, derive new facts from what is given and known. In fact, however, there seems to be another mode of thinking about mathematical concepts, a mode which has little to do with systematic deduction. This other mode is much more difficult to describe and to explain, but it is this special way of thinking which, according to many mathematicians, is the ultimate evidence of deep understanding.

Like the others, ET said explicitly, and more than once, that the ability to construct a proof, or even to use it to construct another argument, does not suffice to give him a sense of a "true" understanding. Here is one of the many remarks he made to that effect:

> I can understand a theorem or a proof at a level that I become convinced about its validity. I can understand a theorem sufficiently to reproduce it in a classroom. All this is still not sufficient evidence for me that I really understand. There is another level, in which I can take a proof of a theorem and prove another theorem with the help of the ideas presented in this proof. Even then, I may still claim that I didn't arrive at a true understanding of the proof.

From different remarks made by the interviewees it was quite clear that, for them, one of the best indications of understanding is the ability to sense that something is true in an immediate manner, without having recourse to a formal proof. This ability to arrive at properties of mathematical objects in a direct way may well be what brought Gauss to make the following statement: "I have had my results for a long time; but I do not know yet how I am to arrive at them" [quoted by Lakatos, 1976, p. 9].

"Having a result" without knowing how it was obtained is perhaps the most striking phenomenon in the work of a mathematician. All my interlocutors have experienced it in the past and they tried to describe it to me in many sometimes quite ingenuous ways. ST used the expression "intimate familiarity" to describe the feeling that accompanies the type of understanding which makes it possible to have direct insight into the properties of mathematical objects. The personification metaphor surfaced again when he tried to explain this special ability to predict the behaviour of abstract constructs:

> When do you feel that you have really understood something? It is only when you are perfectly certain, without having to check, that things must be exactly the way they are. It's like in the case of an intimate familiarity with a person. With such a person you often know what he is going to do without having to ask ... The (abstract) things have a life of their own, but if you understand them, you make predictions and you are pretty sure that you will eventually find whatever you foresaw ... Like a person whom you really know and understand, (the mathematical construct) will perform certain operations or will react in a certain way to your action. This intimacy is exactly what I had in mind: you know what is to happen without making any formal steps. Of course, as in the case of human relationships, you may sometimes be wrong!

The following remark by Johnson [1987] renders the essence of such an "intimate" understanding:

> ... understanding is not only a matter of reflection, using finitary propositions, on some pre-existent, determinate experience. Rather, *understanding is the way we "have a world", the way we experience our world as a comprehensible reality* ... our understanding *is* our mode of "being in the world" ... Our more abstract reflective acts of understanding (which may involve grasping of finitary propositions) are simply an extension of our understanding in this more basic sense of "having a world". [p. 102]

The intimate understanding we are talking about is best explained through a comparison to the way people comprehend basic aspects of the physical world. "'Experiential" comprehension gives people an ability to anticipate behaviors of material objects without reflection. Indeed, when in the blink of an eye we jump to save a leaning glass of water from falling, it is not because we have recalled the law of gravity, confronted it with the empirical data at hand, and made an appropriate inference. Our understanding expresses itself in the ability to know what is going to happen without even being aware of the way in which the prediction was made. Having this kind of understanding endows our method of handling abstract ideas with all the characteristics which, according to Fischbein [1987], are typical of intuitive thinking: our knowledge is self-evident, coercive, global, and extrapolative.

At this point, the central question is what are the sources of this overpowering feeling of obviousness and inevitability about the properties and relations which have not been deductively derived from known facts? How can a mathematician anticipate "behaviors" or abstract structures which have never been seen before? Here, again, the metaphorical nature of mathematical thinking may provide an explanation. One may say that this special mode of reasoning — let us call it *a direct grasp* — becomes possible thanks to the fact that the new mathematical concepts are created in the image of things previously known and of concepts already constructed. It is the carrier of the metaphor — an embodied schema determined by a previous experience — which brings the anticipatory insight. Through the schema, the inner logic and other properties of the new abstract construct are inherited from this earlier experience.

This "hereditary" mechanism which underlies the construction of metaphors has, obviously, some disadvantages. First, because of the experiential origins of the hierarchical sequence of metaphors, the different constraints on our imagination — the basic side-effects of embodiment — are carried like genetic traits from one generation of abstract concepts to another. Some constraints may have to be eased to make the movement toward more abstract ideas possible; nevertheless many of them will be preserved along the way and will continue to delimit mathematical thought.

A second disadvantage has to do with the principle on which the direct grasp is based. Once the abstract objects emerge and their embodied schemata are constructed, our abstract reasoning becomes much like the reasoning induced by sensory perception: it is holistic, immediate, and, above all, it is based on *analogy* rather than on systematic logical inference. The central role of analogy in direct-grasp reasoning was brought to my attention by recurrent references to the "similarity to known facts" made by all my interviewees when they tried to account for their ability to "foresee" the behavior of mathematical objects. The way ET described the mechanism behind his ability to predict mathematical facts is quite typical:

> when you ask me whether something is true or not, I can think about it a moment ... *find a similarity to something else* ... and I can pull an answer out of my sleeve. And all this when I have no inkling about a proof. [my emphasis, A.S.]

It is worth mentioning that the message conveyed by this statement bears a striking similarity to what can be learned from the testimonies of scientists interviewed by Knorr [1980]: "When scientists were asked to tell the story of the origin of a research effort which they considered to be innovative, they regularly displayed themselves as analogical reasoners who build their "innovative" research upon a perceived similarity between hitherto unrelated problems contexts" [p. 31].

Source of the metaphor: a balance between material objects
Target of the metaphor: an equation (an equality between two formulae)

Reasoning:

(1) A fact known from the source domain (material world): a balance between two objects is preserved when the same change in mass is carried on both of them. Let us present it symbolically as a proposition:

P(A, B, CoM)

where A and B are objects and CoM is a change in mass.

(2) metaphorical identification:
- formulae (F, E) are (represent) objects (F = A, E = B)
- equation is (an expression of) a balance between objects
- an operation on a formulae (OoF) is a change of mass of a material object (CoM), namely

  OoF = CoM

(3) Inference:

$$\left.\begin{array}{l} \text{P(A, B, C} \\ \text{F = A, E = B, OoF = CoM} \end{array}\right\} \text{===>} \quad \text{P(F, E, OoF)}$$

So: An equation (equality between two formulae) is preserved if the same operation is performed on both its sides.

Figure 2
Reasoning about equations based
on the metaphor "equality is balance"

The very fact that mathematicians proceed in their work by raising conjectures makes salient their special ability to look ahead and foresee things that are not mere outcomes of logical inference. The following excerpt from Ian Stewart's [1987] account of the way in which André Weil arrived at his famous contributions to the proof of Fermat's Last Theorem underlies, once again, the importance of analogy in mathematical hypothesizing:

> How did Weil come to these conjectures? They weren't just guesswork; he had a strong suspicion that they should hold. They "smelt right". The reason was *analogy* with topology. [p. 33; my emphasis. Stewart's description is based on Weil's own testimonies, A.S.].

The way analogy shapes and curtails mathematical reasoning is presented schematically in Figure 2. The example is rather elementary but it illustrates well the as-if synthetic nature of abstract reasoning based on a metaphor of an experimental, perceptual origin.

Against the claim about a reliance on metaphors some people may maintain that inductive mechanisms play a much more prominent role in mathematicians' thinking than this kind of analogy. The basic principle of mathematical discovery, they would say, is finding the general by scrutiny of the particular. This would imply an intensive use of examples. There can be little doubt that induction does play an important role in mathematical reasoning. There are, however, many facts which speak forcefully against rendering the inductive method exclusivity. First, as we have seen, mathematicians themselves stress their use of analogy. Second, the distinct difficulty some of them experience when trying to explain the sources of their anticipatory abilities seems to indicate that the mechanism of discovery and understanding is less obvious than is implied by the inductive model. Here is a telling excerpt from my conversation with ML, in which the latter says explicitly that "discovery from examples" is definitely not the way he works.

> Examples only confuse me. They don't help me ... I have talked to people in my field who do believe in the power of examples. I am also interested in examples, sometimes. From time to time, I look into some. But in my opinion, if you need examples, it means that you must be quite confused. So how did I arrive at my results? (ML was talking about his specific contribution to model theory.) It's difficult to tell. I think that it took me two years before I arrived at the proper image, and even then I didn't have full proofs for it. I couldn't prove it to other people. But I looked and I saw things. I had a good sense of this world ... I knew what was plausible and what wasn't.

When the mechanism of direct grasp and mathematical invention is concerned, one should mention another important factor which is frequently present in the creative decisions of mathematicians: the criterion of beauty. The role of aesthetic judgment in mathematical reasoning is a recurrent motif in mathematicians' accounts of their own thinking. After stating that "invention is a choice" (and I would say that it is often a choice of metaphor), Hadamard explains that "this choice is imperatively governed by the sense of scientific beauty." [p. 31] Similarly, TS observes:

> I think that there is an element of aesthetics here ... sometimes I make a certain "leap of thought" only because I say to myself that in order for things to be beautiful they must behave in exactly this way and in no other. It must be true this way because otherwise it won't be beautiful enough. I would even go so far as to say that it wouldn't be ethical if it wasn't so

The use of aesthetic criteria, so pervasive in the perceptual domain is, once again, evidence for the metaphorical, embodied character of abstract thinking.

Before I end this section let me remark that the type of thinking and the kind of understanding I present do not have to appear in all creative mathematicians to the same extent. All my interviewees claimed — and, of course, independently — that there exists more than one "kind of mathematical mind". Their remarks may be summarized in a claim that there is a full spectrum of possibilities, at the opposite ends of which stand two basic "styles" of mathematical thinking, styles which may be described as *operational* and *structural*. The operational types have highly developed manipulative skills and use them as a principal means in their quest after meaning. Having a metaphor which makes a mathematical object in the image of a real thing is the dominant need of a structurally-minded mathematician. For the latter type of thinker, the manipulative skills, the ability to draw a systematic argument, are sometimes quite secondary. For example, this was certainly the case with a prominent mathematician, S. Lefschetz, who, according to Halmos [1985] "saw mathematics not as logic but as pictures. His insights were great, but his "proofs" were almost always wrong." [p. 87]

The structuralists are more capable of direct-grasp understanding than those who think and understand in an operational way. This is probably why the belief that structural thinking is superior to operational was implicit in the opinions of the mathematicians I talked to.

To sum up the things that have been said in this section, metaphors impinge upon mathematical reasoning in a very special way: with the emergence of an embodied schema, thought processes may lose their purely analytical character. Metaphorical constructions introduce quasi-synthetical elements into mathematical reasoning. New mathematical truths are no longer discovered through systematic inference from axioms and definitions (are they ever discovered in this way?!); rather, they impose themselves upon a mathematician directly as obvious properties of a mathematical reality. When the abstract construct is supported by an image schema, the perception of its salient characteristics may become much like our perception of the properties of physical bodies: it is immediate, it is holistic, and it is not mediated by a long chain of inferences. It is this ability to grasp ideas in a direct quasi-synthetic way which, according to the mathematicians I talked to, gives them the feeling of "true" understanding. Even though there exist many different kinds of mathematical minds, the phenomenon of direct grasp is probably known to the majority of creative mathematicians.

3.3 Platonism not only for weekdays

In the language introduced by David Hume, the upshot of the last section is that our knowledge of the mathematical realm is not always achieved just by investigating "relations of ideas". Quite often, new truth is discovered (yes, *discovered*) as a "matter of fact". In the eyes of a person who feels that he or she "really" understands an abstract idea, mathematical truth bears a synthetic rather than an analytic character.

"The typical working mathematician is a Platonist on weekdays and a formalist on Sundays", claim Davis and Hersh [1981, p. 321]. From what was said in the previous section it becomes clear that this "practical" Platonism is not a matter of deliberate choice, of insufficient sophistication, or a lack of mathematical (or philosophical) maturity. It is because of the very nature of our imagination, because of our embodied way of thinking about even the most abstract of ideas, that we spontaneously behave and feel like Platonists. Our imagination and reasoning are limited by our sensory experience, and even if we can make a deliberate sortie beyond the constraints of the physical-world lens, such a move, being consciously imposed, may be only temporary. When not forced (by reason) to renounce Platonism on behalf of, say, formalism, our mind will immediately go back to its "natural" state — the state of a Platonic belief in the independent existence of mathematical objects, the nature and properties of which are not a matter of human decision.

Throughout history no new mathematical construct has gained full recognition until mathematicians could feel that, to put it in Davis and Hersh's words, it was as real for them as "the Rock of Gibraltar or Halley's comet". To arrive at such feeling it was not enough to understand the inner logic of a definition and to recognize its consistency with all other mathematical facts. What was necessary was an appropriate metaphor, a metaphor which would show that, in fact, the new idea did not violate the basic laws of the abstract universe. In the Platonic world of ideas the term "basic laws" has a very special meaning and signifies more than the laws of logic. In the realm of material objects all events are determined by laws of nature. Phenomena such as the free fall of a stone thrown from a window are inevitable. Our feeling that the abstract universe is governed by similarly uncompromising, deterministic laws is inherent in the metaphorical way we construct the system of ideas.

The following is a typical confession by a mathematician, talking about one of his major results:

> Mathematicians often argue whether mathematics is discovered or invented. I certainly had the feeling in that particular case that I was discovering it and not inventing it ... We couldn't have invented all that. We had discovered a structure that must have been there. At least, that's the feeling I had; it hung together too well. (Henry Pollak, quoted in Albers and Alexanderson, [1985] p. 243)

Whereas for Pollak the Platonic position might be mainly a matter of a working attitude, some mathematicians openly admit this is their deep philosophical belief. René Thom [1971] is one of them:

> Everything considered, mathematicians should have the courage of their most profound convictions and thus affirm that *mathematical forms indeed have an existence that is independent of the mind considering them.*

In a similar vein, all the mathematicians I talked to spoke about their deep sense of the "reality" of abstract objects, even though some of them emphasized that this was only a matter of feeling and not of sound philosophical belief. This Platonic state of mind, they claimed, was part and parcel of a feeling of deep understanding.

A very interesting contribution to my insight into the relationship between understanding a concept and the belief in its objective existence was provided by ET. ET declared that being a religious person he fully adopts the Platonic view; he then stated that those mathematical concepts which he understands well are conceived by him as referring to objects as real as "a leaf falling from a tree in a forest". For example, he thinks he knows well what an infinite set is, and this feeling of understanding also means that he does "not doubt the existence of an actual infinity". On the other hand, ET does have doubts about real numbers, or rather about the set of all the subsets of the latter. What bothers him is the independence of the continuum hypothesis from the accepted axiomatic systems.

> Until a few years ago I was prepared to declare that our problem with the continuum hypothesis is that we did not formulate (understand) our system in the right way — the way which would make it possible to decide it in this way or another. I could not tolerate the independence of the continuum hypothesis because it was my deep conviction that the set in question must exist or not.

ET's doubt stemmed, obviously, from the unclear status of a certain set with regard to the nature of its existence. His objections aptly substantiate the confining nature of the metaphor. The undecidability of the claim about the existence of an object — any object, either concrete or abstract — defies our basic experientially-based intuitions: it implies that it would be legitimate to assume the existence of a number greater than χ_0 but smaller than χ.

But in our perceptual world, governed by the principle of *tertium non datur*, objects either exist or not and the question "to be or not to be" can only be answered in one way — "yes" or "no". Moreover, it is not up to us to choose the answer. (Incidentally, this example sheds much light on the difficulty mathematicians once had accepting the idea of non-Euclidean geometries.) All this shows the other edge of the metaphorical sword: the constraints which bodily experience puts on our imagination.

Finally, let me mention the enlightening comparisons made independently by LM and by ST between mathematics and chess. The purpose of the comparisons was to show either an objective or just a pragmatic difference between a mere game and an abstraction which is understood and taken seriously. ST, in order to stress the practical importance of a belief in the reality of abstract ideas, shared with me his conviction that "even a chess player, if he is really engaged in the game, cannot think he is just playing ... all this must be for real". ML declared: "When I deal with (mathematical ideas) they exist for me, whether I can justify them philosophically or not". He then explained his position with the following story:

> I was interested in chess. I don't know how able I was as a player, but I stopped doing it at a certain stage. I stopped when I realized I would have to learn it. My feeling was that the game is very interesting, but it is an artificial construction. There is no logical necessity behind these rules. (By contrast) mathematical theories are not arbitrary. You really discover them. You try to move them and after a while you realize that you cannot formulate them in a substantially different way.

In the eyes of LM, the laws of mathematics are "natural" to such extent that one does not have to learn them — they just impose themselves on the mind. The rules of a game, on the other hand, are arbitrary and thus one cannot expect to learn them in a very meaningful way. The interesting question is where these differing perceptions of the nature of the laws of mathematics and the rules of chess come from. The quest for an answer inevitably takes us back to the experiential roots of our imagination: the most plausible explanation for the stance taken by ML is that he had a good working metaphor for his mathematical abstractions while no bodily experience supported the game of chess.

4. Reification as birth of a metaphor

If the meaning of abstract concepts is created through the construction of appropriate metaphors, then metaphors, or figurative projections from the tangible world onto the universe of ideas, are the basis of understanding. As I have already observed in the former sections [see also Sfard, 1987, 1991, 1992], the leading type of sense-rendering metaphor in mathematics is the metaphor of an ontological object. In this last section I will deal very briefly with the intricate question of the way such a metaphor is created and with the inherent difficulties which hinder this process.

I have already discussed these issues quite thoroughly elsewhere. In the present analysis I will try to take advantage of Lakoff and Johnson's theory to both underline certain points I made in the past and to shed a new light on some previously neglected aspects. Out of necessity, I will not go deeply into the subject; in this closing section I will do no more than identify issues for further discussion.

As I once noted [Sfard, 1991], on the face of it there is no reason why we should talk about such impalpable "things" as numbers, functions, sets, groups, and Banach spaces. A closer look at mathematics would reveal that what really counts are processes which we perform mentally, first on physical objects (e.g. counting, measuring), and then, at a higher level, on these primary processes themselves. The fact, however, that the world of abstract mathematical ideas is made in the image of physical reality is in full conformity with Lakoff and Johnson's theory. It is our bodily experience which compels us to think about processes as performed on certain objects and as producing objects. The name "reification" was given to the act of creation of the appropriate abstract entities (some other writers, e.g. Dubinsky [1991], use the term "encapsulation" in a similar way). I may now put it in slightly different words and say that reification is the birth of the metaphor of an ontological object.

The basic claim underlying these ideas is that, from the developmental point of view, operational conceptions precede structural; that is, familiarity with a process is a basis for reification. Using Lakoff and Johnson's ideas I may now broaden the picture and say that, more often than not, reification is the transition from an operational to a structural embodied schema. The classification of schemata into operational and structural requires much more explanation than may be given in this short closing section (Dörfler [1992] and Presmeg [1992] make some slightly different distinctions). Hoping that the ideas are more or less self-explanatory, I will confine myself to a few basic points. An operational schema brings into the domain of abstraction a metaphor of doing, of operating on certain objects to obtain certain other objects. As such, it is a schema of action. The structural embodied schema, on the other hand, conveys a completely different ontological message — a message about a permanent, object-like construct which may be acted upon to produce other constructs. The advantage of the latter type of schema over the former is that it is more integrative, more economical, and manipulable, more amenable to holistic treatment (or parallel processing.) Visual imagery is an integral component.

In the light of the mathematicians' testimonies, the general rule of the developmental precedence of operational conceptions over structural has its exceptions. Mathematicians do not necessarily follow this process-object path. These adepts of abstract thinking, well trained in conjuring new abstract entities out of other abstract entities, may often reach for the metaphor of an ontological object directly, without worrying about the underlying processes. It is certainly the way ST thinks and understands mathematics:

> When I have a new concept, I need a human metaphor. Personification of the concept. Or a spatial metaphor. A new metaphor of a structure. Only when I have it can I answer questions, solve problems, perform manipulations. I can do all this only after I have the metaphor.

Let me stress once more that ST used the word "metaphor" or his own accord, and he heard about the work of Lakoff and Johnson for the first time only after the interview. Notwithstanding his idiosyncrasies, ST suggested (again, of his own accord) that operational-structural periodicity can be detected in many historical processes, such as the development of algebra.

As I have observed many times in the past, reification, whether it precedes or follows the construction of an operational schema, is often achieved only after strenuous effort, if at all. The present treatment of the issue of understanding sheds new light on the inherent difficulty of reification. The frequent problem with new abstract ideas is that they have no counterpart in the physical world or, worse than that, that they may openly contradict our experiential knowledge. Obviously, in the latter case no metaphor is available to support these abstractions. For example, the concept of transfinite numbers violates the fundamental, experientially established principle "the part is less than the whole". This discrepancy between the abstract and the experiential bothered Cantor, the founder of the idea of a transfinite number, to such an extent that he wrote to Dedekind asking for his help in dealing with the thing he himself "could see, but could not believe." In fact, the very idea of reification contradicts our bodily experience: we are talking here about creation of something out of nothing. Or about treating a process as its own product. There is nothing like that in the world of tangible entities, where an object is an "added value" of an action, where processes and objects are separate, ontologically different entities which cannot be substituted one for the other. Our whole nature rebels against the ostensibly parallel idea of, say, regarding a recipe for a cake as the cake itself.

The last remark I wish to make concerns the discontinuous, almost chaotic nature of reification and, more generally, of the process of understanding. A pertinent illustration of what I have in mind here may be found in the excerpt from Halmos' autobiography quoted in the introduction to this paper. Numerous testimonies by mathematicians, including all my interviewees, confirm Hadamard's thesis that sudden illuminations like the one which brought Halmos the "understanding of epsilon" are "absolutely general and common to every student of research" [Hadamard, 1949]. All my interlocutors remarked many times that the process of understanding is full of singularities and sudden jumps. It seems quite likely that the jumps are the result of reification, that they mark the birth of a structural metaphor which gives the concept its "physiognomy" and thereby makes it meaningful. The following two quotations are typical autobiographical stories which aptly illustrate this point. Here is one of them:

> A host of ideas kept surging in my head; I could almost feel them jostling one another, until two of them coalesced, so to speak, to form a stable combination. When morning came, I had established the existence of one class of Fuchsian functions ... I had only to verify the results, which took only a few hours. [Poincaré, 1952, pp. 52-3]

And here is another:

> After struggling for years, the insights eventually came to me that made it all fall into a place. It all hung together in an incredible way — every loose end had its natural location. (Pollak here tells the story of his work on the concentration of signals he did with two other mathematicians; quoted in Albers and Alexanderson [1985] p. 243)

It is remarkable how "physical" is the language used by both Poincare and Pollak in the above descriptions. The mathematicians talk about abstract ideas as if they were material bodies: "It all fell into place" (an expression used also by Halmos: "It all clicked and fell together"!), "it hung together", "they were jostling against each other", "two of them coalesced", "every loose end had its natural location". There can be little doubt that these are stories of a sudden emergence of the metaphor of ontological object.

The issue of discontinuities in the process of understanding seems to be of the utmost importance and, at the same time, it does not yield itself easily to investigation. Freudenthal [1978], who agrees that "*what matters in the learning process are discontinuities*" [p. 165], is nevertheless quite skeptical as to the possibility of empirical research: Discontinuities can only be discovered by continuous observation, but even for teachers and educational researchers it will not be easy to observe these essentials in the learning process. Thus, the thorough study which this "big-bang" phenomenon certainly deserves will have to be preceded by methodological preparations.

5. Morals for experts and for novices

In keeping with Lakoff and Johnson's theory on one hand, and with my own work on the other, I have tried to show in this paper that the metaphor of an ontological object, even though ostensibly only an option in mathematical thinking, is in fact indispensable for the kind of understanding people are prepared to call "deep" or "true". By quoting mathematicians who talked about their own ways of constructing meaning I explained how in this process our bodily experience enters the realm of abstract ideas, both to create it and to confine it. Reification — a transition from an operational to a structural mode of thinking — is a basic phenomenon in the formation of a mathematical concept. Here I have tried to demonstrate that reification is, in fact, the birth of a metaphor which brings a mathematical object into existence and thereby deepens our understanding. The constraints that our perceptually acquired knowledge puts on our imagination make reification inherently difficult.

One conclusion from all that has been said here is that we can educate our imagination and broaden the mathematical universe by loosening perceptual constraints bit by bit, and by gradually paving the way from the mundane to the "never heard of" with an elaborate chain of more and more abstract metaphors. Each layer in the hierarchical edifice of mathematical ideas is a new step in our struggle for freedom from the body restrictions — and for a better understanding of the world of abstraction.

In this paper I have confined myself to mathematicians and to their special ways of struggling for understanding. An important question is to what extent the observations about experts apply also to novices — to school and college students. I have no choice but to leave this question open. I will not finish this paper, however, without formulating some tentative implications for learning and teaching.

The study of mathematicians' ways of thinking brings an important and probably quite universal message about the nature and conditions of understanding. The role of the metaphor of an object in this process cannot be overestimated. Even though the idea may be conveyed in many different disguises, the literature abounds in findings and arguments which support the claim that the natural tendency for structural thinking is typical not only of mathematicians but also of more able students [see e.g. Krutetskii, 1976]. Thus, the immediate implication is that, as teachers, we should foster structural thinking and help "novices" construct their own structural metaphors. The natural question follows: How can we induce the process which brings the metaphor of an object into being? A lot has been said about the inherent difficulty of reification. Studies have shown that even the most sincere efforts to bring about the appropriate metaphor will often be rewarded with only limited success [see e.g. Sfard, 1992]. Because of the tight relationship between the metaphor of an ontological object and the issue of visualization it seems that today's wide accessibility of computer graphics opens promising didactic possibilities.

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Our journalistic sycophants, alas, sometimes offer popularisation which is mere solicitation of popular awe for our recondite mysteries.

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